The Substitution Rule

The **chain rule** greatly expanded the class of functions we could find derivatives of.

The Substitution Rule

The **chain rule** greatly expanded the class of functions we could find derivatives of.

Now we introduce the **substitution rule** for evaluating integrals

The Substitution Rule

The **chain rule** greatly expanded the class of functions we could find derivatives of.

Now we introduce the **substitution rule** for evaluating integrals

The substitution rule applies to integrals of the form

$$\int F'(g(x))g'(x)dx$$

The Substitution Rule

The **chain rule** greatly expanded the class of functions we could find derivatives of.

Now we introduce the **substitution rule** for evaluating integrals

The substitution rule applies to integrals of the form

$$\int F'(g(x))g'(x)dx$$

The result is

$$\int F'(g(x))g'(x)dx = F(g(x)) + C$$

The Substitution Rule

The **chain rule** greatly expanded the class of functions we could find derivatives of.

Now we introduce the **substitution rule** for evaluating integrals

The substitution rule applies to integrals of the form

$$\int F'(g(x))g'(x)dx$$

The result is

$$\int F'(g(x))g'(x)dx = F(g(x)) + C$$

The substitution rule is essentially the chain rule in reverse

Substitution Rule

Example: Evaluate the integral

$$\int 2x \cdot (1 + x^2)^3 dx$$

Substitution Rule

Example: Evaluate the integral

$$\int 2x \cdot (1 + x^2)^3 dx$$

The integrand has the form

$$F'(g(x))g'(x)dx \quad \text{with} \quad F'(u) = u^3 \text{ and } g(x) = 1 + x^2$$

Substitution Rule

Example: Evaluate the integral

$$\int 2x \cdot (1 + x^2)^3 dx$$

The integrand has the form

$$F'(g(x))g'(x)dx \quad \text{with} \quad F'(u) = u^3 \text{ and } g(x) = 1 + x^2$$

Then

$$F'(g(x)) = (1 + x^2)^3 \quad \text{and} \quad g'(x) = 2x$$

Substitution Rule

Example: Evaluate the integral

$$\int 2x \cdot (1+x^2)^3 dx$$

The integrand has the form

$$F'(g(x))g'(x)dx \quad \text{with} \quad F'(u) = u^3 \text{ and } g(x) = 1 + x^2$$

Then

$$F'(g(x)) = (1+x^2)^3 \quad \text{and} \quad g'(x) = 2x$$

Finally

$$F(u) = \frac{u^4}{4} \quad \text{and} \quad \int 2x \cdot (1+x^2) dx = F(g(x)) = \frac{(1+x^2)^4}{4} + C$$

Substitution Rule

Example: Evaluate the integral

$$\int \cos x \sin^2 x dx$$

Substitution Rule

Example: Evaluate the integral

$$\int \cos x \sin^2 x dx$$

The integrand has the form

$$F'(g(x))g'(x)dx \quad \text{with} \quad F'(u) = u^2 \text{ and } g(x) = \sin x$$

Substitution Rule

Example: Evaluate the integral

$$\int \cos x \sin^2 x dx$$

The integrand has the form

$$F'(g(x))g'(x)dx \quad \text{with} \quad F'(u) = u^2 \text{ and } g(x) = \sin x$$

Then

$$F'(g(x)) = \sin^2 x \quad \text{and} \quad g'(x) = \cos x$$

Substitution Rule

Example: Evaluate the integral

$$\int \cos x \sin^2 x dx$$

The integrand has the form

$$F'(g(x))g'((x)dx \quad \text{with} \quad F'(u) = u^2 \text{ and } g(x) = \sin x$$

Then

$$F'(g(x)) = \sin^2 x \quad \text{and} \quad g'(x) = \cos x$$

Finally

$$F(u) = \frac{u^3}{3} \quad \text{and} \quad \int \cos x \sin^2 x dx = F(g(x)) = \frac{\sin^3 x}{3} + C$$

Substitution Rule

Example: Evaluate the integral

$$\int e^{3x} dx$$

Substitution Rule

Example: Evaluate the integral

$$\int e^{3x} dx$$

The integrand has the form

$$F'(g(x))g'(x)dx \quad \text{with} \quad F'(u) = e^u \text{ and } g(x) = 3x$$

Substitution Rule

Example: Evaluate the integral

$$\int e^{3x} dx$$

The integrand has the form

$$F'(g(x))g'(x)dx \quad \text{with} \quad F'(u) = e^u \text{ and } g(x) = 3x$$

Then

$$F'(g(x)) = e^{3x} \quad \text{and} \quad g'(x) = 3$$

Substitution Rule

Example: Evaluate the integral

$$\int e^{3x} dx$$

The integrand has the form

$$F'(g(x))g'(x)dx \quad \text{with} \quad F'(u) = e^u \text{ and } g(x) = 3x$$

Then

$$F'(g(x)) = e^{3x} \quad \text{and} \quad g'(x) = 3$$

Finally

$$F(u) = e^u \quad \text{and} \quad \frac{1}{3}\int 3e^{3x} dx = \frac{1}{3}F(g(x)) = \frac{1}{3}e^{3x} + C$$

Question 1

Find the most general indefinite integral:

$$\int 2 \cdot (1 + 2x)^2 dx$$

1. $\frac{(1+2x)^3}{3} + C$
2. $\frac{(1+2x)^3}{3} + C$
3. $\frac{(1+2x)^3}{3} + C$
4. $\frac{(1+2x)^3}{3} + C$
5. $\frac{(1+2x)^3}{3} + C$
6. None of the choices given

Question 1

Find the most general indefinite integral:

$$\int 2 \cdot (1 + 2x)^2 dx$$

1. $\frac{(1+2x)^3}{3} + C$
2. $\frac{(1+2x)^3}{3} + C$
3. $\frac{(1+2x)^3}{3} + C$
4. $\frac{(1+2x)^3}{3} + C$
5. $\frac{(1+2x)^3}{3} + C$
6. None of the choices given

1. $\frac{(1+2x)^3}{3} + C \quad F'(u) = u^2, \quad g(x) = 1 + 2x$

Question 2

Find the most general indefinite integral:

$$\int \cos 2x dx$$

1. $\frac{\cos x}{2} + C$
2. $\frac{\cos 2x}{2} + C$
3. $\frac{\sin 2x}{2} + C$
4. $\frac{\sin x}{2} + C$
5. $\frac{-\cos 2x}{2} + C$
6. None of the choices given

Question 2

Find the most general indefinite integral:

$$\int \cos 2x dx$$

1. $\frac{\cos x}{2} + C$
2. $\frac{\cos 2x}{2} + C$
3. $\frac{\sin 2x}{2} + C$
4. $\frac{\sin x}{2} + C$
5. $\frac{-\cos 2x}{2} + C$
6. None of the choices given

3 $\quad \frac{\sin 2x}{2} + C \quad F'(u) = \cos u, \quad g(x) = 2x$

Definite Integrals

The substitution rule for evaluating integrals applies to definite integrals as well.

Definite Integrals

The substitution rule for evaluating integrals applies to definite integrals as well.

In this case the integrals have the form

$$\int_a^b F'(g(x))g'(x)dx$$

Definite Integrals

The substitution rule for evaluating integrals applies to definite integrals as well.

In this case the integrals have the form

$$\int_a^b F'(g(x))g'(x)dx$$

The result is

$$\int_a^b F'(g(x))g'(x)dx = \int_{g(a)}^{g(b)} F(u)du$$

Definite Integrals

Example: Evaluate the definite integral

$$\int_0^4 \frac{e^{x/2}}{2} dx$$

Definite Integrals

Example: Evaluate the definite integral

$$\int_0^4 \frac{e^{x/2}}{2} dx$$

The integrand has the form

$$F'(g(x))g'(x)dx \quad \text{with} \quad F'(u) = e^u \text{ and } g(x) = \frac{x}{2}$$

Definite Integrals

Example: Evaluate the definite integral

$$\int_0^4 \frac{e^{x/2}}{2} dx$$

The integrand has the form

$$F'(g(x))g'(x)dx \quad \text{with} \quad F'(u) = e^u \text{ and } g(x) = \frac{x}{2}$$

Then

$$F(u) = e^u \quad g(0) = 0 \quad \text{and} \quad g(4) = 2$$

and the result is

$$\int_0^4 \frac{e^{x/2}}{2} dx = \int_0^2 e^u \, du = \left. e^u \right]_0^2 = e^2 - 1$$

Definite Integrals

Alternatively, one can substitute $g(x) = x/2$ for u and retain the original limits of integration:

$$\int_0^4 \frac{e^{x/2}}{2} dx = \int_0^2 e^u \, du = e^u\Big]_0^2 = e^2 - e^0 = e^2 - 1$$

convert back to a function of x by substituting $x/2$ for u and using the original limits:

$$e^u\Big]_0^2 = e^{x/2}\Big]_0^4 = e^{4/2} - e^{0/2} = e^2 - 1$$

Definite Integrals

Alternatively, one can substitute $g(x) = x/2$ for u and retain the original limits of integration:

$$\int_0^4 \frac{e^{x/2}}{2} dx = \int_0^2 e^u \, du = e^u\Big]_0^2 = e^2 - e^0 = e^2 - 1$$

convert back to a function of x by substituting $x/2$ for u and using the original limits:

$$e^u\Big]_0^2 = e^{x/2}\Big]_0^4 = e^{4/2} - e^{0/2} = e^2 - 1$$

Either method should produce the same result

Question 3

Evaluate the definite integral:

$$\int_0^{\pi} \frac{\sin 4x}{4} dx$$

1. $\int_0^{\pi/2} \sin u du = -\cos u]_0^{\pi/2}$
2. $\int_0^{\pi} \sin u du = -\cos u]_0^{\pi}$
3. $\int_0^{\pi} \cos u du = \sin u]_0^{\pi}$
4. $\int_0^{\pi} \cos u du = -\sin u]_0^{\pi}$
5. $\int_0^{\pi/4} \sin u du = -\cos u]_0^{\pi/4}$
6. None of the choices given

Question 3

Evaluate the definite integral:

$$\int_0^{\pi} \frac{\sin 4x}{4} dx$$

1. $\int_0^{\pi/2} \sin u du = -\cos u]_0^{\pi/2}$
2. $\int_0^{\pi} \sin u du = -\cos u]_0^{\pi}$
3. $\int_0^{\pi} \cos u du = \sin u]_0^{\pi}$
4. $\int_0^{\pi} \cos u du = -\sin u]_0^{\pi}$
5. $\int_0^{\pi/4} \sin u du = -\cos u]_0^{\pi/4}$
6. None of the choices given

5. $\int_0^{\pi/4} \sin u du = -\cos u]_0^{\pi/4}$ = $-\frac{1}{\sqrt{2}} + 1$