The Fundamental Theorem of Calculus

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The reason it is so important is that it relates two seemingly unrelated problems, finding the slope of the tangent line and finding the area under a curve, to the process of differentiation and its reverse, finding an antiderivative.

The theorem allows us to find areas under curves without having to resort to taking limits of Riemann sums.

The Fundamental Theorem of Calculus

The theorem is stated in two separate parts. The first deals with functions defined by an equation of the form

$$g(x) = \int_a^x f(t)dt \quad a \leq x \leq b$$

where f is continuous on $[a, b]$.

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Other than that, t plays no role in evaluating $g(x)$.

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In this case

$$g'(x) = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h}$$

The Fundamental Theorem of Calculus

In this case we can use one of our basic properties of definite integrals, namely

$$\int_a^{x+h} f(t)dt = \int_a^x f(t)dt + \int_x^{x+h} f(t)dt$$

to simplify the expression

$$g(x+h) - g(x) = \int_a^{x+h} f(t)dt - \int_a^x f(t)dt$$

to

$$g(x+h) - g(x) = \int_x^{x+h} f(t)dt$$

The Fundamental Theorem of Calculus

Then the difference quotient

$$\frac{g(x+h) - g(x)}{h}$$

becomes

$$\frac{g(x+h) - g(x)}{h} = \frac{1}{h} \int_x^{x+h} f(t)dt$$

The Fundamental Theorem of Calculus

Because f is continuous on $[x, x+h]$, by the extreme value theorem there are numbers u and v in $[x, x+h]$ at which f attains its absolute minimum $f(u) = m$ and maximum $f(v) = M$ on $[x, x+h]$.

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Another property of definite integrals states (in this case) that if

$$m \leq f(x) \leq M \quad \text{for} \quad x \leq t \leq (x+h)$$

then

$$f(u) \cdot h \;=\; mh \;\leq \int_x^{x+h} f(t)dt \;\leq\; Mh \;=\; f(v) \cdot h$$

The Fundamental Theorem of Calculus

Suppose for the sake of argument that $h > 0$. Then we can divide all terms by h and preserve the inequalities:

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$$x \le u, v \le (x+h)$$

so

$$\lim_{h \to 0} x \le \lim_{h \to 0} u, \lim_{h \to 0} v \le \lim_{h \to 0} (x+h)$$

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Replacing all quantities by their limits, we have

$$x \le \lim_{h \to 0} u, \lim_{h \to 0} v \le x$$

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$$\lim_{h \to 0} u \; = \; x \; = \; \lim_{h \to 0} v$$

Because f is continuous on $[x, x+h]$, we can also say that

$$\lim_{h \to 0} f(u) \; = \; f(x) \; = \; \lim_{h \to 0} f(v)$$

The Fundamental Theorem of Calculus

Now returning to our inequality,

$$f(u) \;\leq\; \frac{1}{h}\int_{x}^{x+h} f(t)dt \;\leq\; f(v)$$

recall that the middle term is equal to

$$\frac{g(x+h)-g(x)}{h}$$

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Substituting this and taking limits as $h \to 0$, we get

$$f(x) = \lim_{h\to 0} f(u) \leq \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \leq \lim_{h\to 0} f(v) = f(x)$$

The Fundamental Theorem of Calculus

Now applying the squeeze theorem,

$$f(x) \leq \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \leq f(x)$$

and so by definition

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = g'(x) = f(x)$$

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This establishes the first part of the Fundamental Theorem of Calculus:

$$\text{if} \quad g(x) = \int_a^x f(t)dt \quad \text{then} \quad g'(x) = f(x)$$

Example 1

If

$$g(x) = \int_1^x t^2 + 3t - 2\ dt$$

find $g'(x)$

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All we need to do is copy it and replace t by x

Example 2

Find

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Again, note that we resisted the temptation to differentiate the integrand, and just copied it replacing t by x.

Question 1

Suppose

$$g(x) = \int_a^x (t^2 - 3t + 2)dt$$

What is $g'(x)$?

1.	$2x - 3$	4.	$t^2 - 3t + 2$
2.	$2t - 3$	5.	$x^2 - 3x + 2$
3.	$2x - 3 + C$	6.	None of the above

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2. $2t - 3$
3. $2x - 3 + C$
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5. $g'(x) = x^2 - 3x + 2$

Question 2

Find

$$\frac{d}{dx}\int_a^x (1+\sinh t)dt$$

1.	$1+\sinh x$	4.	$\cosh x$
2.	$1+\cosh x$	5.	$-\cosh x$
3.	$1+\sinh x+C$	6.	None of the above

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2. $1+\cosh x$
3. $1+\sinh x + C$
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6. None of the above

1. $g'(x) = 1+\sinh x$

The Fundamental Theorem of Calculus

Now for the second form of the Fundamental Theorem.

As with part 1, suppose f is continuous on $[a, b]$. Then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Where F is any antiderivative of f, that is, $F' = f$

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By the second version of the fundamental theorem

$$\int_0^3 x^3 dx \; = \; F(3) - F(0) \quad \text{where} \quad F' = f$$

Using $F(x) = x^4/4$,

$$\int_0^3 x^3 dx \; = \; \frac{3^4}{4} - \frac{0^4}{4} \quad = \quad \frac{81}{4}$$

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$$\int_0^{\frac{\pi}{2}} \sin x dx \;=\; F\left(\frac{\pi}{2}\right) - F(0) \quad \text{where} \quad F' = f$$

Using $F(x) = -\cos x$,

$$\int_0^{\frac{\pi}{2}} \sin x dx \;=\; -\cos\frac{\pi}{2} - (-\cos 0) \quad = \quad 0 - (-1) = 1$$

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$$\int_1^2 x^2 dx$$

1. $\frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$
2. $\frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}$
3. $\frac{2^3}{3} + \frac{1^3}{3} = \frac{9}{3}$
4. $\frac{2^3}{2} - \frac{1^3}{2} = \frac{7}{2}$
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