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An alternative to an algebraic solution is a **numerical** solution.

A numerical solution is a series of computations designed to produce an approximately correct numerical value for a solution to the equation.

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Subsequent approximations are computed by a *recursion* formula, which gives the next approximation as a function of the previous approximations.

For Newton's method, the recursion formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

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Let's examine the recursion formula more closely:

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The equation of the line tangent to the graph of $f(x)$ at $x = x_n$ is:

$$y - f(x_n) = f'(x_n)(x - x_n) \quad \text{or} \quad y = f'(x_n)(x - x_n) + f(x_n)$$

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We obtain the x-intercept (equivalent to a root of the tangent line equation) by setting y to zero:

$$0 = f'(x_n)(x - x_n) + f(x_n)$$

and solving for x

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Now divide both sides by $f'(x_n)$,

$$x - x_n = -\frac{f(x_n)}{f'(x_n)}$$

Finally we add x_n to both sides to get the x intercept as:

$$x = x_n - \frac{f(x_n)}{f'(x_n)}$$

We take this to be x_{n+1} and repeat the process.

Example 1

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In this case the recursion formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2}{3x_n^2}$$

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We can simplify this a bit to

$$x_{n+1} = x_n - \frac{x_n^3 - 2}{3x_n^2} = \frac{2x_n}{3} - \frac{2}{3x_n^2}$$

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and

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Continung,

$$x_5 = \frac{2x_4}{3} - \frac{2}{3x_4^2}$$

We continue until the desired number of correct digits is obtained, which we can tell by how many digits remain the same from the previous iteration.