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- Introduce notation - Assign the quantity to be optimized the symbol Q
- Express Q in terms of the other symbols assigned
- If Q involves a function of more than one variable, eliminate all but one
- Find the **absolute maximum or minimum** of Q

Example 1

Find the point on the line $y = -x + 1$ that is closest to the origin.

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Differentiating with respect to x,

$$2QQ' \quad = \quad 4x - 2 \quad \text{or} \quad Q' = \frac{2x-1}{Q}$$

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The associated y value is $-1/2 + 1$ so the closest point is

$$\left(\frac{1}{2}, \frac{1}{2}\right)$$

Example 2

Suppose we have a set of pairs of x and y values $(x_1, y_1), (x_2, y_2), \ldots$ that can be assumed to fit a model of the form

$$y = bx$$

that is, a model where y is proportional to x.

Example 2

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With real data, the numbers will almost never result in a system of equations that can be solved for b. One approach is to use **least squares**: we choose the value of b that makes

$$Q(b) = \sum_{i=1}^{n} (y_i - bx_i)^2$$

as small as possible (note that we consider Q to be a function of b; The x_i and y_i are constants)

Example 2

Differentiating with respect to b (and using the fact that the derivative of a sum is the sum of the derivatives),

$$\frac{dQ}{db} = \sum_{i=1}^{n} 2(y_i - bx_i) \cdot (-x_i)$$

Example 2

The local max or min will occur at a critical value, so set $Q' = 0$:

$$0 = \sum_{i=1}^{n} 2(y_i - bx_i) \cdot (-x_i) \quad = \quad 2\left(-\sum_{i=1}^{n} x_i y_i + b\sum_{i=1}^{n} x_i^2\right)$$

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Solving for b gives:

$$b_{\mathrm{min}} = \frac{\sum_{i-1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

Example 2

$$b_{\min} = \frac{\sum_{i-1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$$

In statistics, this is known as a **no intercept regression model** and is a commonly used technique for estimating a proportionality constant when the application suggests a model of the form

$$y = bx$$

Example 3

Another example from statistics is the following: Given a set of values $y_1, y_2, \ldots, y_n$, find a value m that makes

$$\sum_{i=1}^{n} (y_i - m)^2$$

as small as possible.

Example 3

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$$\sum_{i=1}^{n} (y_i - m)^2$$

as small as possible.

As before, write the objective function as $Q(m)$:

$$Q(m) = \sum_{i=1}^{n} (y_i - m)^2$$

Example 3

Now differentiate with respect to m and set the result to zero:

$$0 = \sum_{i=1}^{n} 2(y_i - m)(-1) \quad = \quad 2\sum_{i=1}^{n} (m - y_i)$$

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then:

$$0 \quad = \quad \sum_{i=1}^{n}(m - y_i) = nm - \sum_{i=1}^{n} y_i$$

so

$$m_{\min} = \frac{\sum_{i=1}^{n} y_i}{n} = \overline{y}. \quad \text{the mean of } y$$