Stewart Section 4.5 (continued)

Gene Quinn

Indeterminate Forms and l'Hospital's Rule

Theorem(l'Hospital's Rule):

Suppose f and g are differentiable and $g'(x) \neq 0$ for values of x near $x = a$ (with the possible exception of a itself).

Suppose also that the limits as $x \to a$ of f and g are *both* zero or *both* $\pm\infty$.

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$$

or

$$\lim_{x \to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists, or is $\pm\infty$.

l'Hospital's Rule - Things to Keep in Mind

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- It must be the *quotient* of those two functions.
- At the point of interest, the limit must be $0/0$ or $\pm\infty/\pm\infty$.
- Although we have a quotient of functions and we need to take derivatives, *do not* apply the quotient rule. Differentiate the numerator and denominator separately.
- It’s very important to verify that the necessary conditions are met before applying l’Hospital’s rule.

Indeterminate Form Example: $0/0$

Evaluate the limit

$$\lim_{x \to \pi} \frac{\sin x}{x^2 - \pi^2}$$

Indeterminate Form Example: $0/0$

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This is a quotient of functions,

$$\frac{f(x)}{g(x)}$$

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This is a quotient of functions,

$$\frac{f(x)}{g(x)}$$

L'Hospital's rule applies to quotients of functions, so our first step is to evaluate the limits of the numerator and denominator separately,

$$\lim_{x \to \pi} f(x) = \lim_{x \to \pi} \sin x$$

and

$$\lim_{x \to \pi} g(x) = \lim_{x \to \pi} (x^2 - \pi^2)$$

Indeterminate Form Example: 0/0

Both f and g are continuous at π, so we can evaluate the limits by direct substitution:

$$\lim_{x \to \pi} \sin x = \sin \pi = 0$$

and

$$\lim_{x \to \pi} (x^2 - \pi^2) = \pi^2 - \pi^2 = 0$$

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Since the limits of the denominator and numerator are both zero, we have an indeterminate form of type $0/0$, so L'Hospital's rule applies and we can write

$$\lim_{x \to \pi} \frac{f(x)}{g(x)} = \lim_{x \to \pi} \frac{f'(x)}{g'(x)}$$

Indeterminate Form Example: 0/0

Now take derivatives of f and g and evaluate their limits:

$$f'(x) = \frac{d}{dx}\sin x = \cos x$$

$\cos x$ is continuous at $x = \pi$,

so by direct substitution

$$\lim_{x \to \pi} f'(x) = \lim_{x \to \pi} \cos x$$

$$= \cos \pi = -1$$

Indeterminate Form Example: 0/0

$$g'(x) = \frac{d}{dx}(x^2 - \pi^2) = 2x$$

$2x$ is continuous at $x = \pi$,

so by direct substitution

$$\lim_{x \to \pi} g'(x) = \lim_{x \to \pi} 2x$$

$$= 2\pi$$

Indeterminate Form Example: 0/0

L'Hospital's rule says that

$$\lim_{x \to \pi} \frac{f(x)}{g(x)} = \lim_{x \to \pi} \frac{f'(x)}{g'(x)}$$

$$\lim_{x \to \pi} \frac{\cos x}{2x} = \frac{-1}{2\pi}$$

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We now conclude that

$$\lim_{x \to \pi} \frac{\sin x}{x^2 - \pi^2} = -\frac{1}{2\pi}$$

Indeterminate Form Example: 0/0

Example: Find

$$\lim_{\theta \to 0} \frac{\tan \theta}{\ln(1+\theta)}$$

Indeterminate Form Example: 0/0

Example: Find

$$\lim_{\theta \to 0} \frac{\tan \theta}{\ln(1+\theta)}$$

By direct substitution, the limits of the numerator and denominator are both zero:

$$\lim_{\theta \to 0} \tan \theta = \tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

$$\lim_{\theta \to 0} \ln(1+\theta) = \ln(1+0) = 0$$

Indeterminate Form Example: 0/0

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(Answer: 1)

Indeterminate Form Example

Example: Find

$$\lim_{x \to \infty} \frac{\ln x}{x}$$

Indeterminate Form Example

Example: Find

$$\lim_{x \to \infty} \frac{\ln\ x}{x}$$

The numerator

$$\ln\ x$$

and the denominator

$$x$$

go to ∞ as $x \to \infty$.

Indeterminate Form Example

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(Answer: 0)

Indeterminate Form Example

Find

$$\lim_{x \to 0} \frac{\sin\ x}{x \cdot \cos x}$$

Indeterminate Form Example

Find

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Answer: 1

Special Cases

There are a few special cases where the limit is not of the form

$$\frac{0}{0} \quad \text{or} \quad \frac{\pm\infty}{\pm\infty}$$

but can be converted to one of these by careful algebraic manipulation. You then apply l'Hospital's Rule to the converted expression.

The trick in these situations is recognizing how to convert the expression, which may not be obvious.

Special Cases: $0 \cdot \infty$

Suppose you want to find the limit of a product of two functions fg as x approaches some value a, and

$$\lim_{x \to a} f(x) = 0 \quad \text{and} \quad \lim_{x \to a} g(x) = \pm\infty$$

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Write the function fg as

$$fg = \frac{f}{1/g}$$

Then the limit has the form $0/0$ and you can apply l'Hospital's Rule.

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Write the function fg as

$$fg = \frac{f}{1/g}$$

Then the limit has the form $0/0$ and you can apply l'Hospital's Rule.

Alternatively, write the function fg as

$$fg = \frac{g}{1/f}$$

Then the limit has the form $\pm\infty/\pm\infty$ and again you can apply l'Hospital's Rule.

Indeterminate Form Example

Find

$$\lim_{x \to 0^+} \sqrt{x} \ln x$$

Indeterminate Form Example

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Answer: 0

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The limits of the two factors are:

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so the limit of the product has the form $0 \cdot \infty$.

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so the limit of the product has the form $0 \cdot \infty$.

One option is to write the expression as

$$x^2 e^{-x} = \frac{x^2}{e^x}$$

The limit of the new expression has the form $\pm\infty / \pm\infty$, so we can apply l'Hospital's Rule.

Special Cases: $0 \cdot \infty$

The derivative of the numerator is

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and

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The derivative of the denominator is

$$\frac{d}{dx}e^x = e^x$$

and

$$\lim_{x \to \infty} e^x = \infty$$

so the limit of the quotient has the form ∞/∞.

Special Cases: $0 \cdot \infty$

l'Hospital's Rule says that

$$\lim_{x\to\infty} \frac{x^2}{e^x} = \frac{\lim_{x\to\infty} \frac{d}{dx} x^2}{\lim_{x\to\infty} \frac{d}{dx} e^x} = \frac{\lim_{x\to\infty} 2x}{\lim_{x\to\infty} e^x} = \frac{\infty}{\infty}$$

which doesn't seem to help, because we still have ∞/∞.

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which doesn't seem to help, because we still have ∞/∞.

However,

$$\frac{2x}{e^x}$$

satisfies all of the conditions to apply l'Hopital's Rule a second time.

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which doesn't seem to help, because we still have ∞/∞.

However,

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satisfies all of the conditions to apply l'Hopital's Rule a second time.

In fact, it's fairly common to have to apply the rule several times before the resulting quotient has a limit other than $0/0$ or $\pm\infty/\pm\infty$.

Special Cases: $0 \cdot \infty$

On this iteration, we are trying to find the limit of

$$\lim_{x \to \infty} \frac{2x}{e^x}$$

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A second application of l'Hospital's Rule produces

$$\lim_{x\to\infty} \frac{2x}{e^x} = \frac{\lim_{x\to\infty} \frac{d}{dx} 2x}{\lim_{x\to\infty} \frac{d}{dx} e^x} = \frac{\lim_{x\to\infty} 2}{\lim_{x\to\infty} e^x} = 0$$

because the numerator 2 is constant, while the denominator e^x goes to infinity.

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So with two applications of l'Hospital's Rule we are able to conclude that

$$\lim_{x \to \infty} x^2 e^{-x} = 0$$

Special Cases: $\infty - \infty$

Another special case that can sometime be handled is a limit of the form

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Once again, we must be able to convert the difference to a quotient whose limit is $0/0$ or ∞/∞.

Suppose we have

$$\lim_{x \to 1^-} \left(\frac{3}{x-1} - \frac{3x}{x-1} \right)$$

which has the form $\infty - \infty$.

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Suppose we have

$$\lim_{x \to 1^-} \left(\frac{3}{x-1} - \frac{3x}{x-1} \right)$$

which has the form $\infty - \infty$.

Subtract the fractions to obtain

$$\lim_{x \to 1^-} \frac{3-3x}{x-1}$$

which is a $0/0$ form.

Special Cases: $0 \cdot \infty$

l'Hospital's Rule says that

$$\lim_{x\to 1^-} \frac{3-3x}{x-1} = \frac{\lim_{x\to 1^-} \frac{d}{dx} 3-3x}{\lim_{x\to 1^-} \frac{d}{dx} x-1} = \frac{\lim_{x\to 1^-} -3}{\lim_{x\to 1^-} 1} = -3$$

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Of course we could have used cancellation

$$\frac{3-3x}{x-1} = -\frac{x-1}{x-1} = -3$$

to reduce the function to a constant when $x \neq 1$ and it's clear that the limit is -3.

Special Cases: Indeterminate Powers

Limits of the form

$$\lim_{x \to a} [f(x)]^{g(x)}$$

produce indeterminate forms like:

$$0^0 \qquad \infty^0 \qquad \text{and} \quad 1^\infty$$

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Often either taking logarithms, or converting to a product of exponentials will produce a form that allows us to apply l'Hospital's Rule. See the text for examples (p. 304).

Special Cases: Indeterminate Powers

Example: Find

$$\lim_{x \to 0^+} (\cos x)^{1/x}$$

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$$\lim_{x \to 0^+} (\cos x)^{1/x}$$

Solution: This has the form 1^∞. We take the logs of both sides of

$$y = (\cos x)^{1/x}$$

to obtain the $0/0$ form

$$\ln y = \frac{\ln(\cos x)}{x}$$

Applying L'Hospital's rule, we get

$$\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(\cos x)}{x} = \lim_{x \to 0^+} \frac{\frac{1}{\cos x} \cdot (-\sin x)}{1} = 0$$

Special Cases: Indeterminate Powers

Having established that

$$\lim_{x \to 0^+} \ln\,(\cos x)^{1/x} \;=\; 0,$$

we may write

$$\lim_{x \to 0^+} (\cos x)^{1/x} \;=\; e^0 \;=\; 1$$