Stewart Section 4.5

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Indeterminate Forms and l'Hospital's Rule

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$$\lim_{x \to 2} \frac{x-2}{x^2-4}$$

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We've seen that limits of rational functions like this one can be handled by factoring because the numerator and denominator have a common factor ($x-2$ in this case):

$$\lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$$

Indeterminate Forms and l'Hospital's Rule

Although we cannot evaluate

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at $x = 2$, we can evaluate the numerator and denominator separately:

$$(x-2) = 0 \quad \text{when} \quad x = 2$$

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We cannot evaluate the fraction, because when $x = 2$, it has the form

$$\frac{0}{0}$$

which is called an **indeterminate form of type** $\frac{0}{0}$.

Indeterminate Forms and l'Hospital's Rule

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Notice that $f'(2) = 1$ and $g'(2) = 2 \cdot 2 = 4$. Since $g'(2) \neq 0$, the rational function

$$\frac{f'(x)}{g'(x)}$$

is perfectly well-defined at $x = 2$.

Indeterminate Forms and l'Hospital's Rule

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We can find the limits of the numerator and denominator as $x \to 2$:

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Since $g'(2) \neq 0$ we can use direct substitution to obtain

$$\lim_{x\to 2} \frac{f'(x)}{g'(x)} = \frac{\lim_{x\to 2} 1}{\lim_{x\to 2} 2x} = \frac{1}{4}$$

Indeterminate Forms and l'Hospital's Rule

Notice that from the previous result

$$\lim_{x\to 2} \frac{f(x)}{g(x)} = \frac{1}{4}$$

so ast it turns out

$$\lim_{x\to 2} \frac{f(x)}{g(x)} = \lim_{x\to 2} \frac{f'(x)}{g'(x)} = \frac{1}{4}$$

This is not a coincidence, but an example of a theorem called l'Hospital's Rule.

Indeterminate Forms and l'Hospital's Rule

Theorem(l'Hospital's Rule):

Suppose f and g are differentiable and $g'(x) \neq 0$ for values of x near $x = a$ (with the possible exception of a itself).

Suppose also that the limits as $x \to a$ of f and g are *both* zero or *both* $\pm\infty$.

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty$$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists, or is $\pm\infty$.