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As a result, remarkably little information is required to completely characterize the process.

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$$y(t) = y(0)e^{kt} \quad = \quad y(0) \cdot e^{\ln\left(\frac{1}{2}\right)\left(\frac{t}{t_h}\right)} = y(0) \cdot \left(\frac{1}{2}\right)^{\left(\frac{t}{t_h}\right)}$$

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The quantity remaining after 180 days is:

$$y(180) = 300 \cdot \left(\frac{1}{2}\right)^{\left(\frac{180}{140}\right)} \quad = \quad 123.05mg$$

Question 1

bismuth-210 has a half-life of 5 days. How much of a sample of $200mg$ is left after 14 days?

1. $16.48mg$
2. $92.57mg$
3. $54.67mg$
4. $28.71mg$
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Set $y(t)$ equal to $0.9 \cdot y(0)$ and solve for t:

$$y(t) = 0.1 \cdot y(0) = y(0) \cdot \left(\frac{1}{2}\right)^{\left(\frac{t}{t_h}\right)} \Rightarrow 0.1 = e^{\ln\left(\frac{1}{2}\right)\left(\frac{t}{t_h}\right)}$$

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Taking the natrual logarithm of each side gives:

$$\ln 0.1 = \ln\left(\frac{1}{2}\right)\left(\frac{t}{t_h}\right) \quad \text{so}$$

$$t = \frac{t_h \ln 0.1}{\ln\left(\frac{1}{2}\right)} = \frac{140 \cdot \ln 0.1}{\ln\left(\frac{1}{2}\right)} = 465 \text{ days}$$

Question 2

carbon-14 has a half-life of 5730 years. How long does it take for 34% of a sample to decay?

1. $2042 yrs$
2. $6390 yrs$
3. $5531 yrs$
4. $7062 yrs$
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5. $3435 yrs$.

$$t = \frac{5730 \cdot \ln(1 - 0.34)}{\ln\left(\frac{1}{2}\right)} = 3435 \text{ years}$$

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Then the quantity present at time t is:

$$y(t) = y(0)e^{kt} \quad = \quad y(0) \cdot e^{\ln 2\left(\frac{t}{t_d}\right)} = y(0) \cdot (2)^{\left(\frac{t}{t_d}\right)}$$

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The number of cells present after 6 hours is:

$$y(6) = 1000 \cdot (2)^{\left(\frac{6}{2}\right)} \quad = \quad 8000 \text{ cells}$$

Question 3

An experiment with fruit flies finds that the doubling time for the population of Bactrocera zonata on a wheat shorts diet is 2.78 days (Hussain, 1995). If we start with a population of 100 flies, what is the number of flies after 20 days?

1. $14,645$
2. $12,432$
3. $13,167$
4. $15,089$
5. $16,006$
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1. $14,645$.

$$y(20) = 100 \cdot 2^{\frac{20}{2.78}} = 14,645$$

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Set $y(t)$ equal to $10,000$ and solve for t:

$$y(t) = 10000 = 200 \cdot (2)^{\left(\frac{t}{3}\right)} \Rightarrow 50 = e^{\ln(2)\left(\frac{t}{3}\right)}$$

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$$y(t) = 10000 = 200 \cdot (2)^{\left(\frac{t}{3}\right)} \Rightarrow 50 = e^{\ln(2)\left(\frac{t}{3}\right)}$$

Taking the natrual logarithm of each side gives:

$$\ln 50 = \ln(2)\left(\frac{t}{3}\right) \quad \text{so}$$

$$t = \frac{3 \cdot \ln 50}{\ln 2} = 16.93 \text{ hours}$$

Question 4

An experiment with fruit flies finds that the doubling time for the population of Bactrocera zonata on a wheat shorts diet is 2.78 days (Hussain, 1995). If we start with a population of 100 flies, how many days does it take for the population to reach $4,000$ flies?

1. 13.91
2. 15.33
3. 14.79
4. 14.02
5. 16.21
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Question 4

An experiment with fruit flies finds that the doubling time for the population of Bactrocera zonata on a wheat shorts diet is 2.78 days (Hussain, 1995). If we start with a population of 100 flies, how many days does it take for the population to reach $4,000$ flies?

1. 13.91
2. 15.33
3. 14.79
4. 14.02
5. 16.21
6. cannot be determined

3. 14.79.

$$t = \frac{2.78 \cdot \ln 40}{\ln 2} = 14.79 \text{ days}$$

Exponential Cooling

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$$y = T - T_{ambient}$$

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Then

$$y(t) = y(0)e^{kt}$$

Exponential Cooling

Suppose the heat is turned off under a pot of boiling water $(212^\circ F)$ in a room with an ambient temperature of $70^\circ F$. After 10 minutes, the temperature has dropped to $(160^\circ F)$. What is the temperature of the water after 30 minutes?

Exponential Cooling

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Let

$$y = T - T_{ambient}$$

Then

$$y(0) = 212 - 70 = 142$$

and

$$y(10) = 160 - 70 = 90$$

Exponential Cooling

Our exponential cooling equation is then:

$$y(t) = y(0) \cdot e^{kt} \quad \text{so} \quad 90 = 142e^{k \cdot 10} \quad \text{and} \quad \frac{90}{142} = e^{k \cdot 10}$$

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Taking logs of both sides, we get

$$\ln \frac{90}{142} = k \cdot 10 \quad \text{so} \quad k = \frac{\ln \frac{90}{142}}{10} = -0.0456$$

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$$y(t) = 142 \cdot e^{-0.0456t} \quad \text{and} \quad y(30) = 142 \cdot e^{-0.0456 \cdot 30} = 36.15$$

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$$y(t) = 142 \cdot e^{-0.0456t} \quad \text{and} \quad y(30) = 142 \cdot e^{-0.0456\cdot 30} = 36.15$$

So after 30 minutes, the water is $36.15^\circ F$ above ambient, or $106.15^\circ F$.

Question 5

Cooling water from a power plant is pumped into a holding tank at $90°C$. After 20 minutes, the temperature has dropped to $81°C$. If the ambient temperature is $24°C$, what is the temperature of the water after 2 hours?

1. 47.64
2. 51.39
3. 55.21
4. 44.02
5. 46.21
6. cannot be determined

Question 5

Cooling water from a power plant is pumped into a holding tank at $90°C$. After 20 minutes, the temperature has dropped to $81°C$. If the ambient temperature is $24°C$, what is the temperature of the water after 2 hours?

1. 47.64
2. 51.39
3. 55.21
4. 44.02
5. 46.21
6. cannot be determined

2. 51.39. $y(0) = 90 - 24 = 66$ and $y(20) = 81 - 24 = 57$. So

$$\ln \frac{57}{66} = k \cdot 20 \quad \text{so} \quad k = \frac{\ln \frac{57}{66}}{20} = -0.00733$$

then

$$y(120) = 66 \cdot e^{-0.00733 \cdot 120} = 27.39 \quad T_{120} = 27.39 + 24 = 51.39$$