Stewart Section 3.5 - The Chain Rule

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The Chain Rule

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For example, if $f(x) = e^x$, the exponential function rule tells us that

$$\frac{df}{dx} = \frac{d}{dx}e^x = e^x$$

The Chain Rule

Unfortunately, it does not tell us how to handle a function like

$$f(x) = e^{2x},$$

where we have an exponential function not of the independent variable, x, but of some *function* of the independent variable ($2x$ in this case).

The Chain Rule

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This is what makes the chain rule so important - without it, the other rules are somewhat limited in scope.

However, in combination with the chain rule, the rules will allow us to differentiate almost any function we are likely to encounter in an application.

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In the preceding example, if we let

$$g(x) = 2x$$

then our original function is a *composite* function,

$$e^{2x} = f(2x) = f(g(x)) = (f \circ g)(x)$$

with

$$f(x) = e^x \quad \text{and} \quad g(x) = 2x$$

The Chain Rule

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f referred to as the *outer function*, and g is called the *inner function*.

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In this case the outer function f is

$$f(x) = e^x$$

and the inner function g is

$$g(x) = 2x$$

The Chain Rule

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Recall that the notations

$$f(x) = e^x \quad f(u) = e^u \quad f(z) = e^z \quad f(\theta) = e^\theta$$

all describe exactly the same function. The letter symbolizing the independent variable is just a kind of placeholder.

The Chain Rule

$$f(x) = e^x \quad f(u) = e^u \quad f(z) = e^z \quad f(\theta) = e^\theta$$

Since we can use any letter we choose, it is generally considered good form to choose different letters for the independent variables of the inner and outer functions.

Very often, u is used for the outer function, and x for the inner function.

It should be emphasized that we can choose any letter for either role.

The Chain Rule

If we write u for the independent variable of the outer function, the function of our example

$$e^{2x}$$

becomes:

$$f(u) = e^u \quad \text{outer function}$$

$$g(x) = 2x \quad \text{inner function}$$

The Chain Rule - Prime Notation

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If f and g are differentiable, and

$$F = f \circ g$$

is the composite function defined by

$$F(x) = f(g(x))$$

then

$$F'(x) = f'(g(x)) \cdot g'(x)$$

The Chain Rule - Leibnitz Notation

We can also express the chain rule using Leibnitz notation.

If

$$y = f(u)$$

and

$$u = g(x)$$

are differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The Chain Rule - Example

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Using prime notation,

$$F'(x) = f'(g(x)) \cdot g'(x) = f'(2x) \cdot g'(x)$$

The Chain Rule - Example

Now we can apply our exponential rule to

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We apply the rules as well to g:

$$g(x) = 2x$$

$$g'(x) = 2$$

Finally, substitute $2x$ for u in the expression for f' and then

$$F'(x) = f'(g(x)) \cdot g'(x) = f'(2x) \cdot g'(x) = e^{2x} \cdot 2 = 2e^{2x}$$

The Chain Rule - Example

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Then if $y = f(u) = e^u$ and $u = g(x) = 2x$,

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = e^u \cdot 2 = 2e^u = 2e^{2x}$$

The Chain Rule - Step by Step

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Step 1: Identify the inner and outer functions,

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Step 3: Substitute $g(x)$ for u in the expression for $f'(u)$:

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Step 3: Substitute $g(x)$ for u in the expression for $f'(u)$:

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Step 4: Finally, multiply the result by $g'(x)$ to get:

$$(f \circ g)'(x) \quad = \quad f'(g(x)) \cdot g'(x)$$