Stewart Section 2.8

Gene Quinn

The Derivative as a Function

Another way to look at the derivative is to consider it as a function in its own right, called $f'(x)'$, defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

whose domain is the set of points for which the limit exists.

The Derivative as a Function

Another way to look at the derivative is to consider it as a function in its own right, called $f'(x)'$, defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

whose domain is the set of points for which the limit exists.

Definition: A function is said to be **differentiable at** a if $f'(a)$ exists.

A function is **differentiable on an open interval** if it is differentiable at every point in an open interval (that is, an interval of the form (a, b), (a, ∞), $(-\infty, a)$ or $-\infty, \infty)$).

Derivatives

Theorem: If a function f is differentiable at a point a, then f is continuous at a.

Derivatives

Theorem: If a function f is differentiable at a point a, then f is continuous at a.

To prove this theorem, start with a definition of $f'(a)$:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Derivatives

Theorem: If a function f is differentiable at a point a, then f is continuous at a.

To prove this theorem, start with a definition of $f'(a)$:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

As long as $x \neq a$, we can multiply and divide by $x - a$ to get

$$f(x) - f(a) = \frac{f(x) - f(a)}{x - a} \cdot (x - a)$$

Derivatives

Taking the limit of both sides gives

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot (x - a)$$

Derivatives

Taking the limit of both sides gives

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot (x - a)$$

Now applying the product rule for limits, we get

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a)$$

Derivatives

Taking the limit of both sides gives

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot (x - a)$$

Now applying the product rule for limits, we get

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a)$$

The limit as $x \to a$ of $x - a$ is zero, and from the definition of the derivative, we have

$$\lim_{x \to a} [f(x) - f(a)] = f'(a) \cdot 0 = 0$$

Derivatives

Now to complete the proof, note that

$$f(x) = f(a) - [f(x) - f(a)]$$

so

$$\lim_{x \longrightarrow a} f(x) = \lim_{x \longrightarrow a} \{f(a) - [f(x) - f(a)]\}$$

Derivatives

Now to complete the proof, note that

$$f(x) = f(a) - [f(x) - f(a)]$$

so

$$\lim_{x \longrightarrow a} f(x) = \lim_{x \longrightarrow a} \{f(a) - [f(x) - f(a)]\}$$

Now apply the sum rule for limits to get

$$\lim_{x \longrightarrow a} f(x) = \lim_{x \longrightarrow a} f(a) - \lim_{x \longrightarrow a} [f(x) - f(a)]\} = f(a) - 0 = f(a)$$

$/box$

Derivatives

There are essentially three ways a function can fail to be differentiable at a point.

First, the graph of the function can have a sharp corner or kink at $x = a$. In this case, generally the derivative fails to exist because left and right hand limits

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \quad \text{and} \quad \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$$

are not equal.

Derivatives

There are essentially three ways a function can fail to be differentiable at a point.

First, the graph of the function can have a sharp corner or kink at $x = a$. In this case, generally the derivative fails to exist because left and right hand limits

$$\lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} \quad \text{and} \quad \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$$

are not equal.

A simple example of this is the function

$$f(x) = |x|$$

at $a = 0$.

Derivatives

A second way that a function can fail to be differentiable at a point a is for the function to have a jump or discontinuity at $x = a$.

Derivatives

A second way that a function can fail to be differentiable at a point a is for the function to have a jump or discontinuity at $x = a$.

We proved earlier that if f is differentiable at $x = a$, then f is continuous at $x = a$.

The contrapositive of the statement of this theorem is the statement that f is not differentiable at $x = a$ if f is not continuous.

Recall that an if-then statement and its contrapositive are logically equivalent, that is, they always have the same truth value.

Derivatives

The third way that a function can fail to be differentiable at a point a is for the function to have a vertical tangent line (i.e., a vertical asymptote) at $x = a$.

Derivatives

The third way that a function can fail to be differentiable at a point a is for the function to have a vertical tangent line (i.e., a vertical asymptote) at $x = a$.

The function

$$f(x) = \frac{1}{x}$$

is not differentiable at $a = 0$ for this reason.

Derivatives

Having defined the derivative as a function $f'(x)$, there is no reason why we cannot continue to define the derivative of the function $f'(x)$ to obtain the *second derivative* of f denoted by $f''(x)$ and defined by

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

provided the limit exists.

Derivatives

Having defined the derivative as a function $f'(x)$, there is no reason why we cannot continue to define the derivative of the function $f'(x)$ to obtain the *second derivative* of f denoted by $f''(x)$ and defined by

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

provided the limit exists.

In fact, we can continue to define third, fourth, fifth, etc. derivatives of f in the same manner.

Notation

All of the following are common and equivalent notations for the derivative:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called *differentiation operators*.

Notation

All of the following are common and equivalent notations for the derivative:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called *differentiation operators*.

Notations for the second derivative are

$$f''(y) = y'' = \frac{d^2x}{dx^2} = \frac{d^2f}{dx^2} = \frac{d^2}{dx^2}f(x)$$

Notation

All of the following are common and equivalent notations for the derivative:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

The symbols D and d/dx are called *differentiation operators*.

Notations for the second derivative are

$$f''(y) = y'' = \frac{d^2x}{dx^2} = \frac{d^2f}{dx^2} = \frac{d^2}{dx^2}f(x)$$

Notations for the third derivative are

$$f'''(y) = y''' = \frac{d^3x}{dx^3} = \frac{d^3f}{dx^3} = \frac{d^3}{dx^3}f(x)$$