Stewart Section 2.5

Gene Quinn

Limits Involving Infinity

Definition: The notation

$$\lim_{x \to a} f(x) = \infty$$

means that values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a on either side (but not equal to a).

Limits Involving Infinity

Definition: The notation

$$\lim_{x \to a} f(x) = \infty$$

means that values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a on either side (but not equal to a).

An equivalent notation used to describe this situation is

$$f(x) \to \infty \quad \text{as} \quad x \to a$$

Limits Involving Infinity

Definition: The notation

$$\lim_{x \to a} f(x) = \infty$$

means that values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a on either side (but not equal to a).

An equivalent notation used to describe this situation is

$$f(x) \to \infty \quad \text{as} \quad x \to a$$

You will find these phrases used as well, which mean the same thing:

$f(x)$ becomes infinite as x approaches a.

$f(x)$ increases without bound as x approaches a.

Vertical Asymptote

Definition: The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} = \infty \quad \lim_{x\to a^+} = \infty \quad \lim_{x\to a^-} = \infty$$
$$\lim_{x\to a} = -\infty \quad \lim_{x\to a^+} = -\infty \quad \lim_{x\to a^-} = -\infty$$

Vertical Asymptote

Definition: The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to a} = \infty \quad \lim_{x\to a^+} = \infty \quad \lim_{x\to a^-} = \infty$$
$$\lim_{x\to a} = -\infty \quad \lim_{x\to a^+} = -\infty \quad \lim_{x\to a^-} = -\infty$$

Example: The natural logarithm function has a vertical asymptote at $x = 0$ because

$$\lim_{x\to 0^+} \ln x = -\infty$$

Vertical Asymptote

Example: The function

$$y = \frac{1}{x}$$

has a vertical asymptote at $x = 0$ because

$$\lim_{x \to 0^+} f(x) = \infty \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -\infty$$

Vertical Asymptote

Example: The function

$$y = \frac{1}{x}$$

has a vertical asymptote at $x = 0$ because

$$\lim_{x \to 0^+} f(x) = \infty \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -\infty$$

Example: The function

$$f(x) = \frac{3x + 4}{x - 1}$$

has a vertical asymptote at $x = 1$ because

$$\lim_{x \to 1^+} f(x) = \infty \quad \text{and} \quad \lim_{x \to 1^-} f(x) = -\infty$$

Vertical Asymptote

Example: The function

$$y = \frac{1}{(x+1)^2}$$

has a vertical asymptote at $x = -1$ because

$$\lim_{x \to -1^+} f(x) = \infty \quad \text{and} \quad \lim_{x \to -1^-} f(x) = \infty$$

Vertical Asymptote

Example: The function

$$y = \frac{1}{(x+1)^2}$$

has a vertical asymptote at $x = -1$ because

$$\lim_{x \to -1^+} f(x) = \infty \quad \text{and} \quad \lim_{x \to -1^-} f(x) = \infty$$

Example: The function

$$f(x) = \tan(x)$$

has a vertical asymptote at $x = \pi/2$ because

$$\lim_{x \to \pi/2^+} f(x) = \infty \quad \text{and} \quad \lim_{x \to \pi/2^-} f(x) = -\infty$$

Limits at Infinity

Definition: Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} = L$$

means that the value of L can be made as close to L as we like by making x sufficiently large.

Limits at Infinity

Definition: Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} = L$$

means that the value of L can be made as close to L as we like by making x sufficiently large.

An equivalent notation used to describe this situation is

$$f(x) \to L \quad \text{as} \quad x \to \infty$$

Horizontal Asymptote

Definition: The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either:

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

In either case $f(x)$ can be made arbitrarily close to L by taking x sufficiently large in the positive or negative direction.

Horizontal Asymptote

Definition: The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either:

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

In either case $f(x)$ can be made arbitrarily close to L by taking x sufficiently large in the positive or negative direction.

Example: The function

$$f(x) = \frac{1}{x}$$

has a horizontal asymptote at $y = 0$ because

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

Two Special Limits

If n is a positive integer, then

$$\lim_{x \to \infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^n} = 0$$

Two Special Limits

If n is a positive integer, then

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Also

$$\lim_{x\to-\infty} e^x = 0$$

Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the value of $f(x)$ becomes large as x becomes large.

A similar meaning is attached to the symbols:

$$\lim_{x \to -\infty} f(x) = \infty, \quad \lim_{x \to \infty} f(x) = -\infty, \lim_{x \to -\infty} f(x) = -\infty$$

Infinite Limits at Infinity

The notation

$$\lim_{x \to \infty} f(x) = \infty$$

is used to indicate that the value of $f(x)$ becomes large as x becomes large.

A similar meaning is attached to the symbols:

$$\lim_{x \to -\infty} f(x) = \infty, \quad \lim_{x \to \infty} f(x) = -\infty, \lim_{x \to -\infty} f(x) = -\infty$$

Example: The base-e exponential becomes large as x becomes large:

$$\lim_{x \to \infty} e^x = \infty$$

Limits Involving Infinity

Note that ∞ and $-\infty$ are not real numbers.

Limits Involving Infinity

Note that ∞ and $-\infty$ are not real numbers.

It is sometimes useful to consider a set called the **extended real numbers** which is defined to be

$$\mathcal{R} \cup \infty \cup -\infty$$

Certain algebraic properties can be established for the extended real numbers.

$$\infty + \infty = \infty$$

$$-\infty - \infty = -\infty$$

$$\infty + (-\infty) \text{ is undefined}$$

$$\infty \cdot (\pm\infty) = \pm\infty$$

$$\frac{\pm\infty}{\pm\infty} \text{ is undefined}$$