MA125 Stewart Section 2.4

Gene Quinn

Continuity at a Point

Definition: A function f is *continuous at a number* a if

$$\lim_{x \to a} f(x) = f(a)$$

or, equivalently,

$$f(x) \to f(a) \quad \text{as} \quad x \to a$$

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(or, what amounts to the same thing, they are continuous down to a scale that is orders of magnitude smaller than the quantities of interest in the model)

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2) The limit of f as $x \longrightarrow a$ exists:

$$\lim_{x \longrightarrow a} f(x) \quad \text{exists}$$

3) The quantities in conditions 1) and 2) are equal, that is,

$$\lim_{x \longrightarrow a} f(x) = f(a)$$

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This is the same as saying that the function is not defined at $x = a$.

Continuity at a Point

Recall that limit of $f(x)$ as x approaches a requires that the two one-sided limits exist, and that they be equal:

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L \text{ and } \lim_{x \to a^-} f(x) = L$$

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The greatest integer function, defined by

$$[[x]] = \quad \text{the largest integer that is less than or equal to x}$$

is defined for all real numbers, and the left and right hand limits always exist. It fails the test when $x = 2$ because

$$\lim_{x \to 2^+} [[x]] = 2 \quad \neq \quad \lim_{x \to 2^-} [[x]] = 1$$

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Here is an example where 2) holds, but 1) fails: Recall that the function

$$f(x) = \frac{x-1}{x^3-1}$$

is not defined at $x = 1$.

However, the left and right hand limits both exist and are equal,

$$\lim_{x \to 1^+} f(x) = \frac{1}{3} \text{ and } \lim_{x \to 1^-} f(x) = \frac{1}{3}$$

so it passes the second test,

$$\lim_{x \to 1} f(x) = \frac{1}{3} \quad \text{exists}$$

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The simplest way to construct a function that satisfies conditions 1) and 2) but fails condition 3 is to define a function piecewise to make 3) fail.

Starting with the function from the previous example,

$$f(x) = \frac{x-1}{x^3-1}$$

define a function g by

$$g(x) = \begin{cases} f(x) & if \quad x \neq 1 \\ 0 & if \quad x = 1 \end{cases}$$

Continuity at a Point

The function

$$g(x) = \begin{cases} f(x) & if \quad x \neq 1 \\ 0 & if \quad x = 1 \end{cases}$$

is defined at $x = 1$, the left and right hand limits as $x \longrightarrow 1$ exist and are equal, but it fails the third condition because

$$\lim_{x \to 1} f(x) = \frac{1}{3} \neq g(1) = 0$$

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Naturally, this assumes condition 1) holds so that $f(a)$ is defined.

It also assumes a weaker form of condition 2), now stated in terms of only the right hand limit:

$$\lim_{x \to a^+} f(x) = L \quad \text{exists}$$

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A similar concept is defined for left hand limits:

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As before this assumes condition 1) holds so that $f(a)$ is defined.

It also assumes the existence of the left hand limit:

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If the function is not defined outside the interval, the *two-sided* limit fails to exist at the endpoints.

For the puropose of this defining continuity on an interval, we will interpret *continuous* at the endpoint to mean *continuous from the right or continuous from the left*.

Continuity of Sums, Differences, etc.

Determining whether a function is continuous at a point from the definition can involve considerable effort. The following theorem often makes this task much easier:

Theorem: If f and g are continuous at a and c is a constant, then the following functions are also continuous at $x = a$:

$$f + g \quad \text{and} \quad f - g$$

$$cf$$

$$fg \quad \text{and} \quad f/g \quad \text{provided } g(a) \neq 0$$

Continuity of Polynomials and Rational Functions

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b) Any rational function is continuous on its domain.

Recall that the domain of a rational function

$$f(x) = \frac{P(x)}{Q(x)}$$

is all real numbers except for the (real) roots of Q.

Continuity of Classes of Functions

Theorem: The following types of functions are continuous at every point in their domains:

polynomials

rational functions

root functions

trigonometric functions

inverse trigonometric functions

exponential functions

logarithmic functions

Limits of Composite Functions

Theorem: If f is continuous at b and

$$\lim_{x \to a} g(x) = b$$

then

$$\lim_{x \to a} f(g(x)) = f(b)$$

An equivalent statement is:

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

Continuity of Composite Functions

Theorem: If g is continuous at a and f is continuous at $g(a)$, then the composite function

$$(f \circ g)(x) = f(g(x))$$

is continuous at a.

Intermediate Value Theorem

Intermediate Value Theorem: Suppose f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$.

Then there exists a number c in (a, b) such that

$$f(c) = N$$

Intermediate Value Theorem

The intermediate value theorem is intuitively very plausible.

Suppose f represents the temperature between 6:00am and 4:00pm on a given fall day as a function of time that is continuous on the given time interval.

Intuitively, if the temperature was 50 degrees at 6:00am and 72 degrees at 4:00pm, for any temperature value t in the interval

$$50 < t < 72$$

there must have been a point in time between 6:00am and 4:00pm when the temperature was t.

Intermediate Value Theorem

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If this is not true, you have to accept the rather absurd idea that the temperature can jump instantaneously from, say, 59 to 61.

Depending on how the temperature was fluctuating, there could have been more than one point in time when the temperature was 60.

Intermediate Value Theorem

Suppose a particle moves along a straight line for 20 seconds, and its position on the line is given as a function of time by a function $f(t)$ that is continuous on the interval $[0, 20]$:

$$y = f(t) \quad 0 \le t \le 20$$

The particle is at position $f(0)$ on the line at time $t = 0$.

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After 20 seconds, the particle has moved to the position $f(20)$.

The intermediate value theorem in this case says that we can pick any point on the line between $f(0)$ and $f(20)$ and we know that the particle passed through that point at least once on its trip from $f(0)$ to $f(20)$.

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It's also possible that f is not a constant function, but happens to have

$$f(a) = f(b)$$

Intermediate Value Theorem

Loosely speaking, the crucial property that the interval $[a, b]$ has is that there are no “gaps”.

The intermediate value theorem works because if f is continuous on $[a, b]$, there will be no “gaps” in the curve traced by $f(x)$ as x varies from a to b either.