Stewart Section 2.3

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The five limit laws

In the following slides, we assume that c is a constant, and that

$$\lim_{x \to a} f(x) \text{ and } \lim_{x \to a} g(x)$$

both exist.

The five limit laws

1.

$$\lim_{x \longrightarrow a}[f(x) + g(x)] = \lim_{x \longrightarrow a} f(x) + \lim_{x \longrightarrow a} g(x)$$

Which can be stated as “the limit of the sum is the sum of the limits”.

The five limit laws

2.

$$\lim_{x \longrightarrow a}[f(x) - g(x)] = \lim_{x \longrightarrow a} f(x) - \lim_{x \longrightarrow a} g(x)$$

Which can be stated as “the limit of the difference is the difference of the limits”.

The five limit laws

3.

$$\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$$

Which can be stated as “the limit of a constant times a function is the constant times the limit of the function”.

The five limit laws

4.

$$\lim_{x \to a} [f(x) \cdot g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

Which can be stated as “the limit of the product is the product of the limits”.

The five limit laws

5.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if} \quad \lim_{x\to a} g(x) \neq 0$$

Which can be stated as "the limit of the quotient is the quotient of the limits (provided the limit of the denominator is not zero".

Additional limit laws

6. By repeated application of the limit law for products, we obtain

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x) \right]^n$$

Additional limit laws

7. This special limit is used frequently:

$$\lim_{x \to a} c = c$$

Additional limit laws

8. This special limit is also used frequently:

$$\lim_{x \to a} x = a$$

Additional limit laws

9.

$$\lim_{x \to a} x^n = a^n$$

Additional limit laws

10.

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$$

(with the assumption, in the case that n is even, that $a \geq 0$.

Additional limit laws

11.

$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{f(a)}$$

(with the assumption, in the case that n is even, that $f(a) > 0$).

Direct Substitution Property

Definition:

A function f with the property that

$$\lim_{x \longrightarrow a} f(x) = f(a)$$

for every a that belongs to the domain of f is said to have the

direct substitution property

Direct Substitution Property - Polynomials

Theorem:

For any polynomial function

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

the direct substitution property holds for any real number a:

$$\lim_{x \to a} P(x) = P(a) \quad \forall a \in \mathcal{R}$$

Direct Substitution Property - Rational Functions

Theorem:

For any rational function

$$f(x) = \frac{P(x)}{Q(x)} \quad \text{where } P \text{ and } Q \text{ are polynomials}$$

the direct substitution property holds for any real number a in the domain of f:

$$\lim_{x \to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)} \qquad \forall a \text{ such that } Q(a) \neq 0$$

Direct Substitution Property

Polynomials and rational functions are not the only functions that have the direct substitution property.

In Section 2.4, we will see that many other functions have this important property.

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In fact, we will use the direct substitution property to define a class of functions known as *continuous functions*.

Continuous functions are by far the most important class of functions for building mathematical models of real world phenomena.

A Limit Theorem

Theorem:

If f and g are two functions and

$$f(x) = g(x) \quad \text{whenever} \quad x \neq a$$

for some number a, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$$

provided the limits exist.

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No assumptions are made regarding the existence of $f(a)$ and $g(a)$.

Example

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The functions

$$f(x) = \frac{x-1}{x^2-1}$$

and

$$g(x) = \frac{1}{x+1}$$

are identical everywhere except $x = 1$ because for every value of x other than 1,

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The theorem says that if the limits of f and g as $x \longrightarrow 1$ exist, they must be identical, regardless of whether $f(1)$ and $g(1)$ exist.

A Theorem on Inequalities

Theorem

If

$$f(x) \leq g(x)$$

when x is a value near a (with the possible exception of a itself), and

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

both exist, then

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This theorem can be extended to a very useful result known as the *squeeze theorem*.

The Squeeze Theorem

Theorem

If

$$f(x) \leq g(x) \leq h(x)$$

when x is a value near a (except possibly at a itself), and

$$\lim_{x \longrightarrow a} f(x) = \lim_{x \longrightarrow a} h(x) = L$$

then

$$\lim_{x \longrightarrow a} g(x) = L$$

The Squeeze Theorem

The squeeze theorem is useful in the following situation:

- I want to find $\lim_{x \to a} h(x)$ for some function h
- I know of a function g such that $h(x) \leq g(x)$ when x is near a
- I know of a function f such that $f(x) \leq h(x)$ when x is near a
- I can find $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} g(x) = L_2$
- The limits of f and g as $x \to a$ are the same: $L_1 = L_2 = L$

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Now I can use the squeeze theorem to conclude that

$$\lim_{x\to a} h(x) = L$$

without having to actually evaluate this limit.

Sample Problem 1

Given that

$$\lim_{x \to a} f(x) = -3 \quad \text{and} \quad \lim_{x \to a} g(x) = 0$$

evaluate

$$\lim_{x \to a} \frac{f(x) - 3g(x)}{g(x) + f(x)}$$

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$$\lim_{x \to a} \frac{f(x) - 3g(x)}{g(x) + f(x)}$$

By the sum rule, the limit of $g(x) + f(x) = -3 + 0$ is not zero, so by the quotient rule

$$\lim_{x \to a} \frac{f(x) - 3g(x)}{g(x) + f(x)} = \frac{\lim_{x \to a}(f(x) - 3g(x))}{\lim_{x \to a}(g(x) + f(x))}$$

Sample Problem 1

By the sum rule,

$$\frac{\lim_{x\to a}(f(x) - 3g(x))}{\lim_{x\to a}(g(x) + f(x))} = \frac{\lim_{x\to a} f(x) - \lim_{x\to a} 3g(x)}{\lim_{x\to a} g(x) + \lim_{x\to a} f(x)}$$

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By the sum rule,

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Finally by the constant multiple rule,

$$\frac{\lim_{x\to a} f(x) - \lim_{x\to a} 3g(x)}{\lim_{x\to a} g(x) + \lim_{x\to a} f(x)} = \frac{\lim_{x\to a} f(x) - 3\lim_{x\to a} g(x)}{\lim_{x\to a} g(x) + \lim_{x\to a} f(x)}$$

and by substitution

$$= \frac{(-3) - 3 \cdot 0}{0 + (-3)} = \frac{-3}{-3} = 1$$

Sample Problem 2

Evaluate

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Applying the rules for the identity and constant functions (rules 7 and 8), and the rule for a power of a function, we get

$$2(\lim_{x \to 4} x)^2 + 4 - 5$$

Sample Problem 2

Now applying the rule for the identity function $f(x) = x$ again we finally have

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Although it's good practice to use the definitions directly, in this case a good deal of time could be saved by noting that $f(x)$ is a polynomial and 4 is in its domain, so it has the direct substitution property and therefore the limit is

$$f(4) = 2 \cdot 4^2 + 4 - 5 = 31$$

Naturally, on an exam you would want to do the problem this way to save time.

Sample Problem 3

Find

$$\lim_{x\to 1}\left(\frac{1+3x}{1+4x^2+3x^4}\right)^3$$

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The limit can be evaluated directly from the definitions, but it's best in this case to just note that the expression in the limit is a rational function, and 1 is in its domain, so again we can save time and effort by using the direct substitution property to get

$$\left(\lim_{x\to 1}\frac{1+3x}{1+4x^2+3x^4}\right)^3 = \left(\frac{1+3\cdot 1}{1+4\cdot 1^2+3\cdot 1^4}\right)^3 = \left(\frac{4}{8}\right)^3 = \frac{1}{8}$$