Stewart Section 2.2

Gene Quinn

The Limit of a Function

It is often important to describe the behavior of a function f in the neighborhood of some element a of its domain, and what happens to $f(x)$ as x approaches a.

The Limit of a Function

It is often important to describe the behavior of a function f in the neighborhood of some element a of its domain, and what happens to $f(x)$ as x approaches a.

Definition of the limit of a function

We say that *The limit of* $f(x)$ *as* x *approaches* a *equals* L and write

$$\lim_{x \to a} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a, on either side of a.

The Limit of a Function

It is often important to describe the behavior of a function f in the neighborhood of some element a of its domain, and what happens to $f(x)$ as x approaches a.

Definition of the limit of a function

We say that *The limit of* $f(x)$ *as* x *approaches* a *equals* L and write

$$\lim_{x \longrightarrow a} f(x) = L$$

if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a, on either side of a.

The alternative notation

$$f(x) \rightarrow L \quad \text{as} \quad x \rightarrow a$$

has the same meaning as

$$\lim_{x \longrightarrow a} f(x) = L$$

The Limit of a Function

Two aspects of this definition deserve some emphasis.

First, note that x can approach a from the left or the right.

The Limit of a Function

Two aspects of this definition deserve some emphasis.

First, note that x can approach a from the left or the right. We only write

$$\lim_{x \to a} f(x) = L$$

when

$$f(x) \to L$$

regardless of which direction x approaches a from.

The Limit of a Function

Two aspects of this definition deserve some emphasis.

First, note that x can approach a from the left or the right. We only write

$$\lim_{x \to a} f(x) = L$$

when

$$f(x) \to L$$

regardless of which direction x approaches a from.

Second, the value of the function at a, $f(a)$, plays absolutely no role in the definition of the limit.

In fact, a doesn't even have to belong to the domain of f.

One-Sided Limits

A variation of the limit notation

$$\lim_{x \to a} f(x) = L$$

is used to denote the situation where x approaches a from only the left or only the right.

One-Sided Limits

A variation of the limit notation

$$\lim_{x \to a} f(x) = L$$

is used to denote the situation where x approaches a from only the left or only the right.

In the case where x must approach a from the right, we write

$$\lim_{x \to a^+} f(x) = L$$

which is understood to mean that we can make $f(x)$ as close to L as we want by taking values of x close to, but always greater than a

One-Sided Limits

Similarly, if x must approach a from the left, we write

$$\lim_{x \to a^-} f(x) = L$$

which is understood to mean that we can make $f(x)$ as close to L as we want by taking values of x close to, but always less than a

One-Sided Limits

Similarly, if x must approach a from the left, we write

$$\lim_{x \to a^-} f(x) = L$$

which is understood to mean that we can make $f(x)$ as close to L as we want by taking values of x close to, but always less than a

From the definitions of the limit of a function and one-sided limits of a function, we have the important fact that:

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L$$

One-Sided Limits

Put another way, in order for

$$\lim_{x \to a} f(x) = L$$

to exist, *both* one-sided limits

$$\lim_{x \to a^-} f(x) \text{ and } \lim_{x \to a^+} f(x)$$

have to exist, and they have to have the *same* value, L.

Sample Problem

Suppose

$$g(x) = \frac{x-1}{x^3-1}$$

Sample Problem

Suppose

$$g(x) = \frac{x-1}{x^3-1}$$

Recall that the domain of a rational function

$$\frac{P(x)}{Q(x)}$$

is all real numbers except for the (real) roots of $Q(x)$,

In this case, 1 is the only real root of $Q(x)$.

Sample Problem

So, the domain of

$$g(x) = \frac{x-1}{x^3-1}$$

is

$$A = \{x : x^3 - 1 \neq 0\} = R \setminus \{1\}\}$$

which means that $g(x)$ is defined for all real numbers except 1.

Sample Problem

So, the domain of

$$g(x) = \frac{x-1}{x^3-1}$$

is

$$A = \{x : x^3 - 1 \neq 0\} = R \setminus \{1\}\}$$

which means that $g(x)$ is defined for all real numbers except 1.

If you plug in the suggested values listed in the text and evaluate the function, you will see that apparently the left and right hand one-sided limits exist, and they appear to be the same value.

Sample Problem

So, the domain of

$$g(x) = \frac{x-1}{x^3-1}$$

is

$$A = \{x : x^3 - 1 \neq 0\} = R \setminus \{1\}\}$$

which means that $g(x)$ is defined for all real numbers except 1.

If you plug in the suggested values listed in the text and evaluate the function, you will see that apparently the left and right hand one-sided limits exist, and they appear to be the same value.

This numerical experiment does not in any way *prove* that the limit exists, or that it is what it appears to be. Calculator and computer arithmetic is *not* exactly equivalent to real arithmetic.

Sample Problem

Another point about this problem is that the denominator and numerator share a common factor:

$$g(x) = \frac{x-1}{x^3-1} = \frac{x-1}{(x-1)(x^2+x+1)}$$

Sample Problem

Another point about this problem is that the denominator and numerator share a common factor:

$$g(x) = \frac{x-1}{x^3-1} = \frac{x-1}{(x-1)(x^2+x+1)}$$

One might be tempted to “cancel out” the common factors to obtain

$$h(x) = \frac{P_2(x)}{Q_2(x)} = \frac{1}{x^2+x+1}$$

$h(x)$ is a rational function whose denominator $Q_2(x)$ has no real zeros, and therefore has the set of all real numbers as its domain.

Sample Problem

Another point about this problem is that the denominator and numerator share a common factor:

$$g(x) = \frac{x-1}{x^3-1} = \frac{x-1}{(x-1)(x^2+x+1)}$$

One might be tempted to “cancel out” the common factors to obtain

$$h(x) = \frac{P_2(x)}{Q_2(x)} = \frac{1}{x^2+x+1}$$

$h(x)$ is a rational function whose denominator $Q_2(x)$ has no real zeros, and therefore has the set of all real numbers as its domain.

Technically, $h(x)$ and $g(x)$ are *not* the same function: 1 is in the domain of h, but not the domain of g.

Sample Problem

$$g(x) = \frac{x-1}{(x-1)(x^2+x+1)}$$

$$h(x) = \frac{1}{x^2+x+1}$$

It is true that $h(x)$ and $g(x)$ are equal everywhere except $x = 1$, where $h(x) = 1/3$ and $g(x)$ is not defined.

Sample Problem

What *can* be said about $g(x)$ and $h(x)$ is that

$$\lim_{x \to 1} g(x) = \lim_{x \to 1} h(x) = h(1)$$

because the definition of

$$\lim_{x \to 1} g(x)$$

does not require that $g(1)$ exist.