MA125 Stewart Section 1.7

Gene Quinn

Parametric Curves

It is possible to determine a curve by specifying a point (x, y) in terms of *two* functions f and g of a third variable t, called a *parameter*, using the equations

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The position of the particle after t seconds is given by the coordinates

$$(f(t), g(t))$$

Parametric Curves

Example: Suppose we define the following parametric equations:

$$x = \cos(t) \quad \text{and} \quad y = \sin(t) \quad \text{for} \quad t \in (0, \pi)$$

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and moving through some trajectory in the plane.

The trajectory is such that after t seconds, the position of the particle will be

$$(\cos(t), \sin(t))$$

In this case, the parametric curve is the top half of a unit circle centered at the origin.

Parametric Curves

Example: In the specification of a parametric curve,

$$x = f(t), \quad y = g(t)$$

one of the simplest cases occurs when f is the **identity function**,

$$f(x) = x \quad \forall x \in (-\infty, \infty)$$

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In this case, the parametric curve behaves just like the graph of the ordinary function $y = g(x)$.

Example: Define a set of parametric equations with f being the identity function:

$$x = t \quad \text{and} \quad y = 3 * t + 4 \quad \text{for} \quad t \in [0, 4]$$

The parametric curve is identical to the graph of $y = 3x + 4$ on $[0, 4]$.

Parametric Equations

Example: Another a set of parametric equations with f being the identity function:

$$x = t \quad \text{and} \quad y = \sin(t) \quad \text{for} \quad t \in [0, 2\pi]$$

The parametric curve is identical to the graph of $y = \sin x$ on $[0, 2\pi]$.

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Example:

$$x = t \quad \text{and} \quad y = t^2 \quad \text{for} \quad t \in [0, 3]$$

The parametric curve is identical to the graph of $y = x^2$ on $[0, 3]$.

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Example: A slight variation on the previous case can be used to plot the inverse of a function.

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This time, we take g to be the identity function, and let f be the function whose **inverse** we want to plot.

This time, instead of behaving like the graph of $y = f(x)$, the parametric plot will be identical to the graph of $y = f^{-1}(x)$, the inverse of f.

Parametric Equations

Example: As we have seen, if we let f be the identity function, the following system will produce a parametric curve identical to the graph of $y = e^x$ on $[-2, 2]$:

$$x = t \quad \text{and} \quad y = e^t \quad \text{for} \quad t \in [-2, 2]$$

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Example: Now let g be the identity function and $f(x) = e^x$:

$$x = e^t \quad \text{and} \quad y = t \quad \text{for} \quad t \in (-2, 2)$$

The parametric curve is identical to the graph of $y = \ln x$ on $[e^{-2}, e^2]$.

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Often this can be done by taking several values of t, and plotting the points

$$(f(t), g(t))$$

Parametric Curves

Example: Find the parametric curve or trajectory described by the following parametric equations:

$$x = \sin t \quad \text{and} \quad y = \sin^2 t \quad \text{for} \quad t \in [0, 6\pi]$$

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In fact, the trajectory in this case matches the parabola

$$y = x^2 \qquad -1 \leq x \leq 1$$

Parametric Curves

$$x = \sin t \quad \text{and} \quad y = \sin^2 t \quad \text{for} \quad t \in [0, 6\pi]$$

Here is a description of the trajectory for the above parametric equations:

- The particle begins at the origin and traces the parabola to $(1, 1)$
- The particle then reverses direction and traces the parabola back to $(0, 0)$
- The particle then traces the parabola to $(-1, 1)$
- The particle then retraces the parabola back to $(0, 0)$
- The previous 4 steps are repeated twice more, for a total of three times.

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It is possible, and not unusual, for part or all of the trajectory to be traced more than once as t varies over its range of values.

Parametric Equations

Example: The parametric curve produced by

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Example: The parametric curve produced by

$$x = \cos 2t \quad \text{and} \quad y = \sin 2t \quad \text{for} \quad t \in [0, 2\pi]$$

looks identical to the one above, except that the circle is traced **twice** as t varies from 0 to 2π.

Parametric Curves

Note that in the previous two examples, the parametric curve produced by the two functions

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is **not** the graph of a function

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The class of parametric curves is larger than the class of function graphs, because we can always parametrize the graph of $y = f(x)$ by taking

$$x = t \quad \text{and} \quad y = f(t)$$

Converting Parametric to Cartesian

Sometimes it is useful to find an equivalent Cartesian form for a set of parametric equations.

Thas is, given equations of the form

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find y as a function of x alone:

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Usually this is done by elminating t from the equations in the following way:

- use one of the equations to obtain t as a function of x or y
- substitute this function into the other equation wherever t appears

Sample Problem

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Solve the second equation for t:

$$y = 6 - 3t \quad \Rightarrow \quad t = \frac{6-y}{3}$$

Now substitute this into the first equation:

$$x = \left(\frac{6-y}{3}\right)^2 = \frac{1}{9}(6-y)^2$$