MTH125 Stewart Sections 1.1 and 1.2

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Functions - The General Idea

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The idea arises naturally when something “depends on” or “is determined by” something else.

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When an object is released from some height, the distance it has travelled depends on how much time has elapsed since it was released.

In fact, there is a simple formula for the distance d (in feet) that an object falls in t seconds:

$$d = 16t^2$$

Dependency Examples

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At each point in time (starting with $t = 0$), the distance and speed are uniquely determined.

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There are two important points to keep in mind regarding the domain:

First, *every* element of the domain *must* be assigned to an element of the range - no exceptions.

If a function is given by an algebraic formula and the domain is not explicitly stated, the convention is that the domain is assumed to be all real numbers for which the rule of assignment makes sense and produces a real number.

Definition of a Function

Second, no element of the domain can ever be assigned to more than one element of the range. If that happens, f no longer qualifies as a function.

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The set of all possible values of $f(x)$ as x varies through the entire domain is called the *range* of f.
In set notation, the range of f, call it R, is

$$R = \{f(x) : x \in A\}$$

Definition of a Function

A subtle point: Note that in general the range of f, R, is not the same as B, the set whose elements f assigns to the elements of the range, although B must contain R. In set notation, we write $R \subset B$

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In an expression like

$$y = f(x)$$

The symbol y, which represents an arbitrary element of the *range* of f, is called a *dependent variable*.

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The ordered pairs in the graph of f can be associated in the obvious way with the points of the coordinate plane, which gives us a very useful device for describing functions using pictures.

Graph of a Function

The idea of the coordinate plane originated with Rene Descartes in the 17th century, and was one of the great mathematical ideas of that age. Folklore has it that he was inspired by a fly walking across the ceiling.

Graph of a Function

When there is no danger of confusion, we will use the term *graph* to represent both the set

$$G = \{(x, f(x) : x \in A\}$$

and the plot of G in the coordinate plane.

Four Ways to Represent a Function

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verbally	(using a description in words)
numerically	(using a table of values)
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The numerical or table representation is useful for empirical data, which always consists of a finite number of points and usually can't be expressed as a formula.

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Some functions don't lend themselves to visual representation.

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The domain of the absolute value function is all real numbers, and the range is nonnegative real numbers.

Symmetry - Even and Odd Functions

A function f that satisfies

$$f(-x) = f(x)$$

for every x in its domain is called an *even* function.

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A function f that satisfies

$$f(-x) = -f(x)$$

for every x in its domain is called an *odd* function.

The graph of an even function is symmetric about the diagonal line whose equation is $y = x$.

Monotonic Functions

On a given interval I, a function f is said to be *increasing* if

$$f(x_1) < f(x_2) \quad \text{whenever} \quad x_1, x_2 \in I \quad \text{and} \quad x_1 < x_2$$

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The term **monotonic** is used to describe a function that is either increasing or decreasing.

Types of Functions

A *linear* function is a function of the form

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A function is called a *polynomial* if it has the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x^1 + a_0$$

where n is a nonnegative integer and $a_0, 1_1, \ldots, a_n$ are constants.

If the leading coefficient $a_n \neq 0$ the *degree* of the polynomial is n.

Polynomials of degree 1, 2, and 3 are called linear, quadratic, and cubic, respectively.

Types of Functions

A *power* function is a function of the form

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If a is a positive integer, then f is a one-term polynomial of degree a.

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$$f(x) = \frac{\sqrt{1-x}}{\sqrt{1+x}}$$

and

$$\sqrt{3x^3 - 7x + 1}$$

are algebraic functions.

Rational Functions

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The domain of a rational function consists of all values of x for which $Q(x) \neq 0$.

Trigonometric, Exponential, and Logar

A *trigonometric* function is one of the functions:

$$\sin x, \;\; \cos x, \;\; \tan x$$

and their reciprocals

$$\csc x, \;\; \sec x, \;\; \cot x$$

Exponential and Logarithmic Function

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A *logarithmic* function is a function of the form

$$f(x) = \log_a x$$

where the base a is a positive constant.

Transcendental Functions

A *trancendental* function is one that is not algebraic.

Trancendental functions include the trigonometric functions and their inverses, exponential and logarithmic functions, as well as a large class of functions without names defined by infinite series.

An Example of a Mathematical Model

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One of the radical ideas presented was that the mass of an object increases when it is moving, and in fact is a function of the speed or velocity with which the object is moving.

Mathematical Model Continued

Einstein proposed that the mass of an object is a function of its velocity v, with the following rule of assignment:

$$mass = f(v) = \frac{m_0}{\sqrt{1 - (v^2/c^2)}}$$

where m_0 is the mass of the object at rest, and c is the speed of light.

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As the value of v approaches c from the left, or $-c$ from the right, what happens to the function value $f(v)$?

What physical interpretation can you give to this property of the mathematical model?