The Derivative as a Function

Because the letter we use to represent the independent variable does not matter, replacing a by x in the definition of the derivative of a function $f(x)$ gives

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The domain of the derivative f' may be the same as the domain of f or smaller, but cannot be larger.

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A continuous function fails to be differentiable anywhere it has a sharp corner, or a vertical tangent.

Example 1

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as usual we make use of the conjugate technique:

$$\lim_{x \to a} \frac{\sqrt{x-1} - \sqrt{a-1}}{x-a} \left(\frac{\sqrt{x-1} + \sqrt{a-1}}{\sqrt{x-1} + \sqrt{a-1}} \right)$$

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The domain of f' is $(1, \infty)$.

We have to exclude 1, although it is in the domain of f.

The Second Derivative

If we regard the derivative f' as a function in its own right, there is no reason why we cannot repeat the process of finding the derivative, except this time starting with f':

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} \quad \text{if the limit exists}$$

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In this case f'' represents (instantaneous) *acceleration*.

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$$f''(x) = \lim_{x \to a} \frac{f'(x) - f'(a)}{x - a} = \lim_{x \to a} \frac{\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{a}}}{x - a}$$

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$$= \lim_{x \to a} \frac{\frac{2(\sqrt{a} - \sqrt{x})}{2\sqrt{x}2\sqrt{a}}}{x - a} = \lim_{x \to a} \frac{\frac{\sqrt{a} - \sqrt{x}}{2\sqrt{x}\sqrt{a}}}{x - a} \left(\frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right)$$

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This simplifies to

$$f''(a) = \lim_{x \to a} = \frac{-1}{2\sqrt{x}\sqrt{a}(\sqrt{x} + \sqrt{a})} = \frac{-1}{4(\sqrt{a})^3}$$

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$$f(t) = 4000 - 16t^2$$

Find the instantaneous *acceleration* at $t = 2$.

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This simplifies to

$$f'(a) = \lim_{x \to a} \frac{-16(x^2 - a^2)}{x - a} = \lim_{x \to a} -16(x + a) = -32a$$

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Now we find the instantaneous acceleration, which is the second derivative

$$f''(a) = \lim_{x \to a} \frac{f'(x) - f'(a)}{x - a} = \lim_{x \to a} \frac{-32x - (-32a)}{x - a}$$

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So the instantaneous acceleration at time t is given by $A(t) = -32$, that is, the constant function whose value is -32 for every value of t.