MA125 Exam 3 Version 1

Name:

1) Find two positive numbers x and y such that $x + 4y = 1000$ and the product xy is as large as possible.

Solution: We need to write the product xy as a function of one variable, so solve for x:

$$x = 1000 - 4y$$

Then

$$Q = P(y) = xy = (1000 - 4y) \cdot y = 1000y - 4y^2$$

Now we differentiate with respect to y to get

$$P'(y) = 1000 - 8y$$

Setting this to zero and solving, we get

$$1000 = 8y \quad \text{and so} \quad y = 125 \quad \text{and} \quad x = 1000 - 4y = 500$$

2) If we are using Newton's method to find a critical number of

$$f(x) = e^x - \sin x$$

with $x_0 = 1$, what is the value of x_2?

Newton's method is used to find solutions to an equation of the form $f(x) = 0$. In this case, we are asked to find a critical number of $f(x) = e^x - \sin x$, which means a value of x that satisfies:

$$f'(x) = e^x - \cos x = 0$$

So for this problem, $f'(x)$ will play the role of f, and our recursion formula is:

$$x_{n+1} = x_n - \frac{f'(x)}{f''(x)}$$

or

$$x_{n+1} = x_n - \frac{e^x - \cos x}{e^x + \sin x}$$

We are to use $x_0 = 1$, so our first iteration (with $n = 0$) will compute x_1:

$$x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)} = 1 - \frac{e - \cos 1}{e + \sin 1} = 1 - \frac{2.178}{3.560} = 0.3882$$

Now the second iteration computes x_2:

$$x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)} = 0.3882 - \frac{e^{0.3882} - \cos 0.3882}{e^{0.3882} + \sin 0.3882} = 0.388 - \frac{0.549}{1.853} = 0.09203$$

3) Find the absolute maximum and minimum of the function

$$f(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 3x + 2 \quad \text{on} \quad [-2, 1]$$

To find the absolute maximum and minimum of f we use the closed interval method: Find all critical numbers of f in $[-2, 1]$, then evaluate f at each of these values, plus the two endpoints -2 and 1. The smallest of these values is the absolute minimum, the largest is the absolute maximum.

First we find the critical numbers by solving $f'(x) = 0$:

$$f'(x) = 0 = -x^2 + x + 3$$

The quadratic does not factor, so we have to use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{1 - 4(-1)(3)}}{-2} = \frac{-1 \pm \sqrt{13}}{-2}$$

The critical numbers are -1.3028 and 2.3028, but only the first one is inside the interval $[-2, 1]$, so we ignore the second and compute

$$f(-1.3028) = -1.797$$

Now we compute

$$f(-2) = 0.6667 \quad \text{and} \quad f(1) = 5.1667$$

So the three candidates for absolute maximum and minimum are:

x	$f(x)$
-1.3028	-1.797
-2	0.6667
1	5.1667

The absolute minimum of 0.08635 occurs at $x = -0.8028$, and the absolute maximum of 5.1667 occurs at the right endpoint, $x = 1$.

4) Find the point on the curve

$$f(x) = 1 + \sqrt{x}$$

that is closest to $(1, 1)$.

The distance from $(1, 1)$ to (x, y) is

$$D^2 = (x - 1)^2 + (y - 1)^2$$

Since $y = 1 + \sqrt{x}$, we can substitute to obtain

$$D^2 = (x - 1)^2 + (1 + \sqrt{x} - 1)^2 = x^2 - 2x + 1 + (\sqrt{x})^2 = x^2 - x + 1$$

Now we have the distance D as a function of a single variable x, so we can differentiate to obtain

$$2DD' = 2x - 1 \quad \text{and} \quad D' = \frac{2x - 1}{2D}$$

Setting $D' = 0$ and solving for x gives:

$$2x - 1 = 0 \quad \text{so} \quad 2x = 1$$

and $x = 1/2$. Then

$$(x, y) = \left(\frac{1}{2}, 1 + \sqrt{\frac{1}{2}}\right) = (0.5, 1.707)$$

5) Find

$$\lim_{x \to 0} \frac{e^{4x} - 1 - 4x}{x^2}$$

6) An object initially at rest is released from a height of $3000ft$ and falls with acceleration function

$$a(t) = -32$$

Find the position of the object after 10 seconds.

Solution: The position function $s(t)$ is an antiderivative of the velocity function $v(t)$, which in turn is an antiderivative of the acceleration function $a(t)$. Since $a(t)$ is constant, the general antiderivative is:

$$v(t) = -32t + C$$

Where C is to be chosen to make the initial velocity $v(0)$ equal to zero, because we are given that the object is initially at rest. So

$$v(t) = -32t + 0 = -32t$$

Now we find the position function as the general antiderivative of $v(t)$,

$$s(t) = -32\left(\frac{t^2}{2}\right) + C = -16t^2 + C$$

Now we choose C to make $s(0)$ match the position at time $t = 0$, which is 3000 feet:

$$s(0) = -16 \cdot 0^2 + C = 3000$$

so $C = 3000$, and the position at time t is:

$$s(t) = -16t^2 + 3000$$

and the position after 10 seconds is:

$$s(10) = -16(10^2) + 3000 = -1600 + 3000 = 1400 \text{ feet}$$

7a) If $f(x) = x^2 - 4x + 1$, which (if any) of the following conclusions can we draw from the Mean Value Theorem?

a) $f'(c) = 6$ for some $c \in (4, 6)$
b) $f'(c) = 4$ for some $c \in (4, 6)$
c) $f'(c) = 2$ for some $c \in (4, 6)$
d) $f'(c) = 0$ for some $c \in (4, 6)$
e) None of the above, the theorem does not apply

Solution: The Mean Value Theorem states that if f is continuous on some closed interval $[a, b]$, then there is a $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

If we consider the interval $[4,6]$, f is continuous on this interval (because polynomials are continuous everywhere) and so we may conclude that there exists a $c \in (4,6)$ such that

$$f'(c) = \frac{f(6) - f(4)}{6-4} = \frac{13-1}{6-4} = 6$$

so the only possible answer is (a).

7b) If $f(x) = x^2 - 5x + 6$, which (if any) of the following conclusions can we draw from Rolle's Theorem?

a) $f'(c) = 0$ for some $c \in (0,2)$
b) $f'(c) = 0$ for some $c \in (1,4)$
c) $f'(c) = 0$ for some $c \in (2,3)$
d) $f'(c) = 0$ for some $c \in (-2,0)$
e) None of the above

Solution: Rolle's Theorem states that if f is continuous on some closed interval $[a,b]$ and $f(a) - f(b)$, then there is a $c \in (a,b)$ such that

$$f'(c) = 0$$

In order for the theorem to apply, the function values have to be the same at both endpoints. For the first choice, $f(0) = 6$ and $f(2) = 0$, so the theorem does not apply. For the second choice, $f(1) = 2$ and $f(4) = 2$, so the theorem applies. It also applies for choice (c) because $f(2) = 0$ and $f(3) = 0$. so possible answers are (b) and (c) (either one is acceptable; actually both are true) .

7c) If $f'(x) = 0$ for $x \in (0,3)$, which of the following conclusions can be drawn?

a) $f''(2) > 0$
b) $f(1) = 0$
c) $f(3) - f(0) > 0$
d) $f(1) = f(2)$
e) None of the above

Solution: We have a theorem that states that if $f'(x) = 0$ everywhere in an interval I, then f is constant on I. So we conclude that $f(1) = f(2)$. Answer: (e).

7d) If $f'(x) = g'(x)$ for $x \in (-3, 3)$, which of the following conclusions can be drawn?

a) $f'(-3) < 0$
b) $f(-3) = g(-3)$
c) $f(2) - g(2) = f(0) - g(0)$
d) $f(0) = g(0)$
e) None of the above

Solution: We have a theorem that states that if $f'(x) = g'(x)$ everywhere in an interval I, then f and g differ by a constant on I. So we conclude that $f(2) - g(2) = f(0) - g(0)$. Answer: (c).

7e) If f has a local minimum at $x = 3$, what does Fermat's theorem say about $f'(3)$?

a) $f'(3) = 0$
b) if $f'(3)$ exists then $f'(3) = 0$
c) $f'(3) > 0$
d) $f'(3) < 0$
e) Either $f'(3) = 0$ or $f'(3)$ does not exist

Solution: Fermat's theorem states that if f has a local minimum or maximum at 3, then if $f'(3)$ exists it must be zero. Answer: Depending on how you interpreted the question, (a), (b), or (e) would be acceptable. Credit was given for any of these three.

Suppose

$$f(x) = -\frac{x^3}{3} - x^2 + 3x + 7$$

8a) Which of the following lists contains **all** of the intervals on which f is increasing?

a) $(-\infty, -1)$
b) $(-\infty, -3), (1, \infty)$
c) $(-3, 1)$
d) $(-1, \infty)$
e) $(-\infty, \infty)$

Solution: f is increasing on any interval where $f'(x) > 0$. Since $f'(x) = -x^2 - 2x + 3 = (1 - x)(3 + x)$ is zero at -3 and 1, and its graph is a parabola opening down, f' is positive on $(-3, 1)$. Answer: (c)

8b) Which of the following lists contains **all** of the intervals on which f is decreasing?

a) $(-\infty, -1)$
b) $(-\infty, -3), (1, \infty)$
c) $(-3, 1)$
d) $(-1, \infty)$
e) $(-\infty, \infty)$

Solution: Since $f'(x) = -x^2 - 2x + 3 = (1 - x)(3 + x)$ is zero at -3 and 1, and its graph is a parabola opening down, f' is negative on $(-\infty, -3)$ and $(1, \infty)$. Answer: (b)

8c) Which of the following lists contains **all** of the intervals on which f is concave up?

a) $(-\infty, -1)$
b) $(-\infty, -3), (1, \infty)$
c) $(-3, 1)$
d) $(-1, \infty)$
e) $(-\infty, \infty)$

Solution: f is concave up on any interval where $f''(x) > 0$. Since $f''(x) = -2x - 2$ is zero at $x = -1$, and its graph is a straight line with slope -2, f'' is positive to the left of -1, on $(-\infty, -1)$. Answer: (a)

8d) Which of the following lists contains **all** of the intervals on which f is concave down?

a) $(-\infty, -1)$
b) $(-\infty, -3), (1, \infty)$
c) $(-3, 1)$
d) $(-1, \infty)$
e) $(-\infty, \infty)$

Solution: f is concave down on any interval where $f''(x) < 0$. Since $f''(x) = -2x - 2$ is zero at $x = -1$, and its graph is a straight line with slope -2, f'' is negative to the right of -1, on $(-1, \infty)$. Answer: (d)

8e) Which of the following lists contains **all** of the intervals on which f' is increasing?

a) $(-\infty, -1)$
b) $(-\infty, -3), (1, \infty)$
c) $(-3, 1)$
d) $(-1, \infty)$
e) $(-\infty, \infty)$

Solution: f' is increasing on any interval where $f''(x) > 0$. Since $f'(x) = -2x - 2$ is zero at -1 and positive to the left of -1, f' is increasing on $(-\infty, -1)$. Answer: (a)

9) Find the most general antiderivative of:

$$g(x) = \frac{5 - 4x^3 + 2x^6}{x^6}$$

Solution: We do not yet have techniques for finding antiderivatives of rational functions, but we can convert to a sum of powers of x:

$$g(x) = \frac{5 - 4x^3 + 2x^6}{x^6} = \frac{5}{x^6} - \frac{4x^3}{x^6} + 2$$

or

$$g(x) = 5x^{-6} - 4x^{-3} + 2$$

Now applying the rule for antiderivatives of powers of x, we have

$$G(x) = \frac{5x^{-5}}{-5} - \frac{4x^{-2}}{-2} + 2x + C = -x^{-5} + 2x^{-2} + 2x + C$$

10) Find the limit:

$$\lim_{x \to -\infty} x^2 e^x$$

Solution: Note that

$$x^2 \to \infty \quad \text{and} \quad e^x \to 0 \quad \text{as} \quad x \to -\infty$$

so the product has the form $0 \cdot \infty$. We have to convert it to $0/0$ or ∞/∞, which we can do by rewriting the expression as:

$$\lim_{x \to -\infty} \frac{x^2}{e^{-x}}$$

Now the numerator and denominator both go to ∞ as $x \to -\infty$ so it has the form ∞/∞ and we can apply L'Hospital's rule:

$$\lim_{x \to -\infty} \frac{x^2}{e^{-x}} = \lim_{x \to -\infty} \frac{2x}{-e^{-x}}$$

The limit is now $-\infty/-\infty$, so we can apply L'Hospital's rule a second time:

$$\lim_{x \to -\infty} \frac{2x}{-e^{-x}} = \lim_{x \to -\infty} \frac{2}{e^{-x}} = 0$$

because the numerator is constant, while the denominator tends to infinity.