MA125 Exam 2 Version 1

Name:

1) The rate of change of the *difference* between the temperature of an object and the ambient temperature is proportional to that difference. A gallon of tap water is drawn from a faucet at $48°F$ and let stand in a room with an ambient temperature of $75°F$. After 15 minutes, the temperature of the water has risen to $55°F$. What will the temperature of the water be after 1 hour?

Solution: By Newton's las of cooling, the rate of change *difference* between the temperature of the water and the ambient temperature, $D = T - T_{ambient}$ is proportional to that difference.

Initially, the difference is:

$$D(0) = T(0) - T_{ambient} = 48 - 75 = -27$$

After 15 minutes, the difference is

$$D(15) = T(15) - T_{ambient} = 55 - 75 = -20$$

The equation relating $D(0)$ and $D(15)$ is:

$$D(15) = D(0)e^{15k} \quad \text{or} \quad -20 = -27 \cdot e^{15k}$$

so

$$\ln\left(\frac{-20}{-27}\right) = 15k \quad \text{or} \quad k = \frac{\ln(-20/-27)}{15} = -0.02$$

and the temperature difference at time t is:

$$D(t) = -27e^{-0.02t} \quad \text{so} \quad D(60) = -27e^{-0.02 \cdot 60} = -8.13$$

so

$$T(60) = T_{ambient} + D(60) = 75 - 8.13 = 66.87$$

2) Find the equation of the line tangent to the graph of

$$x^2 + 2xy - y^2 + x = 2 \quad \text{at} \quad (1, 2)$$

Solution: Differentating implicitly, we have

$$2x + (2y + 2xy') - 2yy' + 1 = 0$$

We can solve this for y and substitute in $x = 1$ and $y = 2$, but alternatively we could substitute for x and y first:

$$2 \cdot 1 + (2 \cdot 2 + 2 \cdot 1 \cdot y') - 2 \cdot 2y' + 1 = 0 \quad \text{or} \quad 2 + 4 + 2y' - 4y' + 1 = 0$$

so

$$7 - 2y' = 0 \quad \text{and} \quad y' = \frac{7}{2}$$

The equation of the tangent line is

$$y - f(1) = f'(1) \cdot (x - 1) \quad \text{or} \quad y - 2 = \frac{7}{2}(x - 1)$$

which can be simplified to

$$y = \frac{7}{2}x - \frac{3}{2}$$

3) Drugs are eliminated from the bloodstream in such a way that the rate of change of the concentration of a drug is proportional to its concentration. A patient recovering from surgery is given a dose of the painkiller dihydromorphinone at midnight that produces an initial concentration of 0.5mg/l. Two hours later, the concentration is 0.3mg/l. If the next dose must be given when the concentration reaches 0.1mg/l, what time should the next dose be given?

Solution: The fact that the rate of change of the concentration of the drug is proportional to the concentration tells us that the exponential growth/decay model applies:

$$C(t) = C(0)e^{kt}$$

The given information states that $C(0) = 0.5$ and $C(2) = 0.3$, so we can determine k by

$$0.3 = 0.5e^{2k} \quad \text{so} \quad \ln\left(\frac{0.3}{0.5}\right) = 2k$$

and

$$k = \frac{\ln(0.3/0.5)}{2} = -0.2554$$

so

$$C(t) = 0.5e^{-0.2554t} \quad \text{and} \quad 0.1 = 0.5e^{-0.2554t}$$

$$t = \frac{\ln(0.1/0.5)}{-0.2554} = 6.302$$

so the next dose should be given at about $6:18AM$.

4) A particle moves along the curve $y = \sqrt{x^2 - 2}$. The x-coordinate of the particle is increasing at a rate of 5cm/sec as it reaches the point $(3, \sqrt{7})$. How fast is the distance from the particle to the origin changing at that instant?

Solution: The square of the distance from the origin to a point on the curve is:

$$D^2 = (x-0)^2 + (\sqrt{x^2-2} - 0)^2 \quad = \quad 2x^2 - 2$$

Differentiating implicitly,

$$2D\frac{dD}{dt} = 4x\frac{dx}{dt}$$

substituting $dx/dt = 5$,

$$\frac{dD}{dt} = \frac{5 \cdot 4x}{2D}$$

When $x = 3$ and $y = \sqrt{7}$,

$$D = \sqrt{x^2 + y^2} = \sqrt{3^2 + \sqrt{7}^2} = \sqrt{16} = 4$$

so

$$\frac{dD}{dt} = \frac{5 \cdot 4 \cdot 3}{2 \cdot 4} = \frac{15}{2}$$

5) A hailstone grows in size as it falls through humid air. Assuming it has the shape of a sphere whose volume is increasing at the rate of $0.4\text{cm}^3/\text{min}$, how fast is its radius changing when the volume is 1.3cm^3?

Solution: We need to express the radius of a sphere as a function of its volume. The usual formula relating the two is

$$V = \frac{4}{3}\pi r^3$$

solving for r gives

$$r = \sqrt[3]{\frac{3V}{4\pi}} = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}}$$

differentiating with respect to time,

$$\frac{dr}{dt} = \frac{1}{3}\left(\frac{3V}{4\pi}\right)^{-\frac{2}{3}}\left(\frac{3}{4\pi}\right)\frac{dV}{dt}$$

substituting $V = 1.3$ and $dV/dt = 0.4$ gives

$$\frac{dr}{dt} = \frac{1}{3}\left(\frac{3 \cdot 1.3}{4\pi}\right)^{-\frac{2}{3}}\left(\frac{3}{4\pi}\right) \cdot 0.4 \;= = 0.2257$$

6) The pressure P, temperature T, volume V of an ideal gas are related by the gas law:

$$PV = nRT$$

where n is the number of moles of gas present and $R = 0.0821$ is the gas constant. If 3 moles of an ideal gas are heated at a rate of 2 kelvins/minute in a sealed 5 liter container, what is the rate of change of the pressure P?

Since the container is sealed, the volume V is constant at $5l$, we have

$$P = \left(\frac{nR}{V}\right) T$$

Differentiating with respect to time,

$$\frac{dP}{dt} = \left(\frac{nR}{V}\right)\frac{dT}{dt} = \left(\frac{3 \cdot 0.0821}{5}\right) \cdot 2 \;=\; 0.098$$

7) An object undergoing *damped oscillation* has position function

$$s(t) = Ae^{-ct}\cos(\omega t + \delta)$$

where A, c, ω, and δ are fixed. Find the velocity $v(t)$ and acceleration $a(t)$ of the object.

Solution: The velocity is the derivative of the position function $s(t)$ with respect to t. Applying the product rule gives:

$$v(t) = s'(t) = -Ace^{-ct}\cos(\omega t + \delta) - Ae^{-ct}\omega\sin(\omega t + \delta)$$

The acceleration is $a(t) = v'(t) = s''(t)$, which is obtained by differentiating $v(t)$ with the product rule:

$$a(t) = Ac^2e^{-ct}\cos(\omega t + \delta) + 2Ace^{-ct}\omega\sin(\omega t + \delta) - Ae^{-ct}\omega^2\cos(\omega t + \delta)$$

8) Find the derivative of

$$f(t) = \cos^2\left(e^{\sinh t}\right)$$

This is a multi-level chain rule problem:

$f(u) = u^2$
$g(v) = \cos v$
$h(w) = e^w$
$j(t) = \sinh t$

So the derivative of $f(g(h(j(t)))) = \cos^2\left(e^{\sinh t}\right)$ has the form

$$f'(g(h(j(t)))) \cdot g'(h(j(t)) \cdot h'(j(t)) \cdot j'(t)$$

or

$$\left(2\cos e^{\sinh t}\right) \cdot \left(-\sin e^{\sinh t}\right) \cdot \left(e^{\sinh t}\right) \cdot (\cosh t)$$

9) The cost of producing x yards of a certain fabric is:

$$C(x) = 1200 + 12x - 0.1x^2 + 0.0005x^3$$

Find the marginal cost of producing 200 units.

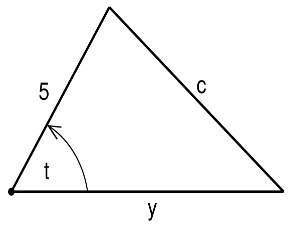
Solution: The marginal cost of producing 200 units is $C'(200)$, where

$$C'(x) = 12 - 0.2x + 0.0015x^2$$

so

$$C'(200) = 12 - 0.2(200) + 0.0015(200)^2 = 32$$

10) One side of a tringle has a fixed length of 5**cm** and the base y is increasing at a rate of 3cm/min. The angle t is increasing at the rate of 0.2rad/min. How fast is the length of side c changing when $y = 5$ and $t = \pi/6$?



Solution: The law of cosines applied to this problem gives the equation:

$$c^2 = 5^2 + y^2 - 2 \cdot 5y \cos t$$

Note that in this case, t and y are variables, so we need to use the product rule to differentiate:

$$2cc' = 0 + 2yy' - 10(y' \cos t - y \sin t)$$

so

$$c' = \frac{yy' - 5y' \cos t + 5yt' \sin t}{c}$$

where

$$y' = 3\text{cm/min}, \quad t' = 0.2\text{rad/min}, \quad y = 5, \quad \text{and} \quad t = \pi/6$$

Then

$$\cos t = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} = 0.866 \quad \text{and} \quad \sin t = \sin \frac{\pi}{6} = 0.5$$

and, again using the law of cosines, we compute c as:

$$c = \sqrt{5^2 + 5^2 - 2 \cdot 5 \cdot 5 \cdot 0.866} = 2.588$$

so

$$c' = \frac{5 \cdot 3 - 5 \cdot 3 \cdot 0.866 + 5 \cdot 5 \cdot 0.2 \cdot 0.5}{2.588} = 1.742\text{cm/min}$$

Formulas:

Volume of a sphere	$V = \frac{4}{3}\pi r^3$
Volume of a cylinder	$V = \pi r^2 h$
Volume of a cone	$V = \frac{1}{3}\pi r^2 h$
Surface area of a sphere	$A = 4\pi r^2$
Hyperbolic functions	$\sinh x = \frac{e^x - e^{-x}}{2}$ $\quad \cosh x = \frac{e^x + e^{-x}}{2}$
Pythagorean theorem	$c^2 = a^2 + b^2$
Law of cosines	$c^2 = a^2 + b^2 - 2ab\cos C$