MA125 Exam 1 Version 1

Name:

1) Use the **definition of the derivative as a limit of a difference quotient** to find the derivative of the following function:

$$f(x) = \frac{1}{x^2}$$

(*Do not* use any formulas for a derivative such as of a power of x or others)

Solution. *By definition,*

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{\frac{1}{x^2} - \frac{1}{a^2}}{x - a}$$

Using the formula

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

we get

$$= \lim_{x \to a} \frac{\frac{a^2 - x^2}{x^2 a^2}}{x - a} = \lim_{x \to a} \frac{a^2 - x^2}{x^2 a^2 (x - a)} = \lim_{x \to a} - \frac{x^2 - a^2}{x^2 a^2 (x - a)}$$

$$= \lim_{x \to a} - \frac{(x - a)(x + a)}{x^2 a^2 (x - a)} = \lim_{x \to a} - \frac{x + a}{x^2 a^2}$$

by direct substitution, the limit is

$$- \frac{2a}{a^4} = \frac{-2}{a^3}$$

(which agrees with the result of nx^{n-1} for $n = -2$). Alternatively, this could have been written as

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

Again using the formula for a difference of two fractions, we get we get

$$= \lim_{h \to 0} \frac{\frac{x^2 - (x+h)^2}{(x+h)^2 x^2}}{h} = \lim_{x \to a} \frac{x^2 - x^2 - 2xh - h^2}{h(x + h)^2 x^2} = \lim_{h \to 0} - \frac{-2xh - h^2}{h(x + h)^2 x^2}$$

If $h \neq 0$, we can cancel the common factors of h in the numerator and denominator to get

$$\lim_{h\to 0} -\frac{-2x-h}{(x+h)^2x^2}$$

and by direct substitution the limit is

$$\frac{-2x}{x^4} = \frac{-2}{x^3}$$

2) Find the equation of the line tangent to the graph of

$$f(x) = -x^2 + 2x + 4$$

at $x = 2$.

Solution. *The equation of the tangent line at $x = 2$ is:*

$$y - f(2) = f'(2)(x - 2)$$

Using the differentiation formulas, the derivative is

$f'(x) = -2x+2$ *so* $f'(2) = -2(2)+2 = -2$ *and* $f(2) = -2^2+2\cdot 2+4 = 4$

so the tangent line is

$$y - 4 = -2(x-2) \quad or \quad y = -2x + 8$$

3) An object is dropped from the top of the CN tower in Toronto. The position of the object t seconds after release is given by

$$f(t) = 1815 - 16t^2$$

Determine the value of a for which the *instantaneous* velocity $v_{inst}(t)$ at time $t = a$ is the same as the *average* velocity $v_{avg}(t)$ from time $t = 0$ to time $t = a$.

Solution. *The average velocity from $t = 0$ to $t = a$ is the difference quotient*

$$\frac{f(a) - f(0)}{a - 0} = \frac{(1815 - 16a^2) - (1815 - 16 \cdot 0^2)}{a - 0} = \frac{-16a^2}{a} = -16a$$

The instantaneous velocity at $t = a$ is $f'(a)$. By the differentiation formulas, $f'(x) = -32x$ so $f'(a) = -32a$. Equating the instantaneous and average velocities, we get

$$-16a = -32a \quad or \quad 0 = -16a \quad so \quad a = 0$$

That is, the instantaneous and average velocities are only equal at the time of release.

4) Suppose $g(x)$ satisfies the following inequalities for all $x \in \mathbb{R}$:

$$\frac{\sqrt{x+6} - x}{x^3 - 3x^2} \leq g(x) \leq \frac{x^2 - 11x + 24}{cx^3(x-3)}$$

Find the value of c that will allow us to apply the squeeze theorem, then apply it to find $\lim_{x\to 3} g(x)$

Solution. *In order for the squeeze theorem to work, the limits of the leftmost and rightmost expressions as $x \to 3$ must be the same. On the left, the limit is*

$$\lim_{x\to 3} \frac{\sqrt{x+6} - x}{x^3 - 3x^2} = \lim_{x\to 3} \frac{\sqrt{x+6} - x}{x^3 - 3x^2} \left(\frac{\sqrt{x+6} + x}{\sqrt{x+6} + x} \right)$$

$$= \lim_{x\to 3} \frac{x + 6 - x^2}{x^2(x-3)} = \lim_{x\to 3} -\frac{(x-3)(x+2)}{x^2(x-3)(\sqrt{x+6} + x)} = \lim_{x\to 3} -\frac{(x+2)}{x^2(\sqrt{x+6} + x)}$$

by direct substitution, this is $-5/54$. The right hand limit is

$$\lim_{x\to 3} \frac{x^2 - 11x + 24}{cx^3(x-3)} = \lim_{x\to 3} \frac{(x-3)(x-8)}{cx^3(x-3)} = \lim_{x\to 3} \frac{x-8}{cx^3}$$

Again by direct substitution this is $-5/27c$. Equating the results, we have

$$-\frac{5}{54} = -\frac{5}{27c} \quad so \quad c = 2$$

5) At a temperature of $15°C$ the volume V of one mole of carbon dioxide is related to its pressure P by the formula

$$V = \frac{1}{P}$$

Find the average rate of change of the volume V per unit change in pressure when the pressure varies over the interval from 0.5 to 1, and the instantaneous rate of change of volume per unit change in pressure when the pressure is 1.

Solution. *The average rate of change of volume from $P = 0.5$ to $P = 1$ is the difference quotient*

$$\frac{V(1) - V(.5)}{1 - .5} = \frac{\frac{1}{1} - \frac{1}{.5}}{1 - .5} = \frac{1 - 2}{.5} = -2$$

The instantaneous rate of change at $P = 1$ is the derivative

$$V'(P) = \frac{d}{dP}\frac{1}{P} = \frac{d}{dP}P^{-1} = -P^{-2} = -\frac{1}{P^2} = -\frac{1}{1^2} = -1$$

6) Suppose C is a positive constant. Determine whether the following limit exists. If it exists, find its value.

$$\lim_{x \to -\infty} \left(3 + \sqrt{x^2 + C} - \sqrt{x^2 - C}\right)$$

Solution. *Write the limit as*

$$\lim_{x \to infty} 3 + \lim_{x \to infty} \left(\sqrt{x^2 + C} - \sqrt{x^2 - C}\right)$$

and use the conjugate to simplify

$$= 3 + \lim_{x \to -\infty} \left(\sqrt{x^2 + C} - \sqrt{x^2 - C}\right) \left(\frac{\sqrt{x^2 + C} + \sqrt{x^2 - C}}{\sqrt{x^2 + C} + \sqrt{x^2 - C}}\right)$$

$$= 3 + \lim_{x \to -\infty} \frac{2C}{\sqrt{x^2 + C} + \sqrt{x^2 - C}}$$

Since $\sqrt{x} \to \infty$ as $x \to \infty$, the denominator becomes large without bound as $x \to \infty$ but the numerator is constant, so the limit of the fraction as $x \to \infty$ is zero, and the final result is

$$\lim_{x \to -\infty} \left(3 + \sqrt{x^2 + C} - \sqrt{x^2 - C}\right) \quad = \quad 3$$

7 Suppose c is a positive constant. For what values of c does the Intermediate Value Theorem guarantee that

$$f(x) = x^3 - c$$

assumes a value of zero somewhere in the interior of the interval $[0, 1]$?

Solution. *The function is continuous on* $[0, 1]$*, so the Intermediate Value Theorem states that it will assume every value in the interval* $(f(0), f(1))$*, or in this case* $(-c, 1 - c)$*. So the theorem guarantees a root as long as this interval contains zero, which it will as long as*

$$-c < 0 \quad \textit{and} \quad 1 - c > 0$$

The first inequality holds when $c > 0$*, and second when* $c < 1$*, so the values of* c *that satisfy both inequalities are*

$$0 < c < 1$$

Any value of c *in this range guarantees a root in the interval* $(0, 1)$*.*

8 A function $f(x)$ is defined piecewise by the following rule of assignment, where b is a positive constant:

$$f(x) = \begin{cases} \frac{1-x^2}{4x+4} & \textit{when} \quad x < -1 \\ \frac{x^2}{x(x-1)} & \textit{when} \quad -1 \le x < 1 \\ (x-1)^2 & \textit{when} \quad 1 \le x < 2 \\ 1 + b^{x-2} & \textit{when} \quad x \ge 2 \end{cases}$$

Which of the following statements are true and which are false?

T		$\lim_{x\to -1^-} f(x)$ exists
T		$\lim_{x\to -1^+} f(x)$ exists
T		$f(x)$ is continuous at $x = -1$
T		$f(x)$ is continuous from the right at $x = -1$
T		$\lim_{x\to 0} f(x)$ exists
	F	$f(x)$ is continuous at $x = 0$
	F	$f(x)$ is continuous from the right at $x = 0$
	F	$\lim_{x\to 1^-} f(x)$ exists
T		$\lim_{x\to 1^+} f(x)$ exists
T		$f(x)$ is continuous from the right at $x = 1$
T		$\lim_{x\to 2^-} f(x)$ exists
T		$\lim_{x\to 2^+} f(x)$ exists
	F	$\lim_{x\to 2} f(x)$ exists
	F	$f(x)$ is continuous from the left at $x = 2$
T		$f(x)$ is continuous from the right at $x = 2$
T		$f(x)$ is continuous at $x = 3$

The first part of the function, for $x < -1$, is

$$f_1(x) = \frac{1-x^2}{4x+4} = \frac{(1-x)(1+x)}{4(1+x)} = \frac{1-x}{4} \text{ except at } x = -1$$

So

$$\lim_{x\to -1^-} f_1(x) = \lim_{x\to -1^-} \frac{1-x}{4} = \frac{1}{2}$$

by direct substitution. The second part of the function, for $-1 \leq x < 1$, is

$$f_2(x) = \frac{x^2}{x(x-1)} = \frac{x}{x-1} \text{ except at } x = 0$$

So

$$\lim_{x \to -1^+} f_2(x) = \lim_{x \to -1^+} \frac{x}{1-x} = \frac{1}{2} = f_2(-1)$$

by direct substitution. Since the left and right hand limits exist at $x = -1$, the two-sided limit exits. The function value exists, and the function value matches the limits, so f is continuous at $x = -1$. Recall that a function is continuous at a point if and only if it is both left and right continuous there, so f is also right continuous at $x = -1$.

The limit of

$$\frac{x}{x-1}$$

as $x \to 0$ is zero, and since f_2 equals this function everywhere except zero, the limit of f_2 is also zero, so

$$\lim_{x \to 0} f(x) \quad \text{exists}$$

However, f_2 is not defined at zero, so the function is neither continuous nor right continuous at $x = 0$.