MA125 Exam 2

Name: KEY

1) Given two functions f and g and a real number a such that

$$f(a) = 0 \quad f'(a) = 1 \quad g(a) = 2$$

find the **y-intercept** of the line tangent to the quotient function

$$y = \left(\frac{f}{g}\right)(x) \qquad \text{at } x = a$$

Solution: We have to apply the quotient rule to evaluate the derivative of f/g at $x = a$. We know the result will be

$$\left.\left(\frac{f}{g}\right)'\right|_{x=a} = \frac{g(a) \cdot f'(a) - f(a) \cdot g'(a)}{g(a)^2}$$

$$= \frac{2 \cdot 1 - 0 \cdot g'(a)}{2^2} = \frac{1}{2}$$

The equation of the line tangent to f/g at $x = a$ is:

$$y - \left(\frac{f}{g}\right)(a) = \left(\frac{f}{g}\right)'(a) \cdot (x - a)$$

Using the fact that $(f/g)(a) = f(a)/g(a) = 0$ and $(f/g)'(a) = 1/2$, by substitution the equation of the tangent line is

$$y - 0 = \frac{1}{2} \cdot (x - a)$$

or

$$y = \frac{1}{2}x - \frac{1}{2}a$$

2 An object is launched from the ground at an angle. The vertical position (y-coordinate) and horizontal position (x-coordinate) after t seconds are given by the functions:

$$x(t) = 50t \quad \text{and} \quad y(t) = 160t - 16t^2$$

a) What is the *horizontal* velocity at $t = 1$?
b) What is the *vertical* acceleration at $t = 3$?
c) At what time(s), if any, is the *vertical* velocity zero?

d) At the instant the maximum y-coordinate is reached, what is the x-coordinate?
e) What is the **average** *vertical* velocity from $t = 0$ to $t = 4$?

Solution: a) The horizontal velocity is the time rate of change of x,

$$\frac{dx(t)}{dt} = \frac{d}{dt}50t = 50$$

b) The vertical acceleration is the second derivative of $x(t)$,

$$\frac{d^2y(t)}{dt^2} = \frac{d}{dt}(160 - 32t) = -32$$

c) The vertical velocity is zero when

$$v_y(t) = \frac{d}{dt}y(t) = 160 - 32t = 0$$

this happens when $t = 160/32 = 5$.
d) From c), the maximum y coordinate is reached at $t = 5$, so the x-coordinate is $x(5) = 50 \cdot 5 = 250$.
e) The average vertical velocity is the change in position from $t = 0$ to $t = 4$ divided by the change in time $(4 - 0)$,

$$V_{avg} = \frac{y(4) - y(0)}{4 - 0} = \frac{(160 \cdot 4 - 16 \cdot 4^2) - (160 \cdot 0 - 16 \cdot 0^2)}{4}$$

$$= \frac{640 - 256}{4} = \frac{384}{4} = 96$$

3 The volume and radius of a sphere are related by the formula:

$$V = \frac{4}{3}\pi r^3$$

Air is being added to a spherical balloon in such a way that the compression is negligible.
a) What is the rate of change of the *radius* with respect to the *volume* when the balloon contains 1000 cm^3 of air?
b) What is the average rate of change of the radius as the volume changes from 100 to 400 cm^3?

Solution: a) If we consider the radius to be a function of the volume and differentiate with respect to V,

$$\frac{d}{dV}V = \frac{d}{dV}\frac{4}{3}\pi r^3$$

so

$$1 = \frac{4}{3} \cdot 3\pi r^2 \cdot \frac{dr}{dV}$$

and therefore

$$\frac{1}{4\pi r^2} = \frac{dr}{dV}$$

To find the value of dr/dV when $V = 1000$ cm^3, we need to find the radius corresponding to $V = 1000$. Solving the volume formula for r gives:

$$r = \sqrt[3]{\frac{3V}{4\pi}} = \sqrt[3]{30004}\pi = \sqrt[3]{238.7} = 6.20$$

so when $V = 1000$,

$$\frac{dr}{dV} = \frac{1}{4\pi \cdot (6.20)^2} = \frac{1}{483.6} = 0.00207$$

b) The average rate of change of the radius, considered as a function of volume, is the total change in the radius $r(V)$, divided by the total change in the volume V:

$$\Delta r_{avg} = \frac{r(V_2) - r(V_1)}{V_2 - V_1}$$

As before we need to convert the volumes to the corresponding values of r:

$$V_2 = 400 \qquad r(V_2) = \sqrt[3]{\frac{3 \cdot 400}{4\pi}} = \sqrt[3]{95.49} = 4.57$$

$$V_1 = 100 \qquad r(V_1) = \sqrt[3]{\frac{3 \cdot 100}{4\pi}} = \sqrt[3]{23.87} = 2.88$$

so the average rate of change of the radius is:

$$\Delta r_{avg} = \frac{4.57 - 2.88}{400 - 100} = \frac{1.69}{300} = .00564$$

4 Find the equation of the line tangent to the curve defined implicitly by

$$x^2y + 2xy^2 + 3x = y$$

at the origin.

Solution: Differentiating implicitly with respect to x gives

$$2xy + x^2yy' + 2y^2 + 2x(2yy') + 3 = y' \quad \text{or} \quad 2xy + x^2yy' + 4xyy' + 3 = y'$$

As usual we collect terms containing y' on one side of the equation,

$$2xy + 3 = (1 - x^2y - 4xy)y' \quad \text{so} \quad y' = \frac{2xy + 3}{1 - x^2y - 4xy}$$

At the origin $x = y = 0$, so this reduces to $y' = 3$. To find the tangent line, start with the usual formula

$$y - f(a) = f'(a)(x - a)$$

In this case $a = 0$, $f(a) = 0$, and $f'(a) = 3$, so the equation of the tangent line is:

$$y = 3x$$

5 At noon ship A is 150km west of ship B. Ship A is sailing east at 35km/hr and ship B is sailing north at 25km/hr. How fast is the distance between the ships changing at 4pm?

Solution: There are a number of ways to solve this problem. One way is to assign the ship's positions to a coordinate system tha places ship B at the origin at noon. The coordinates of ship A at noon are then $(-150, 0)$. With this setup, the coordinates of the ships after t hours are:

$$\text{Ship A: } (35t - 150, 0) \qquad \text{Ship B: } (0, 25t)$$

Let the x-coordinate of ship A at time t be $A(t) = 35t - 150$ (the y-coordinate is always zero for ship A), and let the y-coordinate of ship B at time t be $B(t) = 25t$ (the x-coordinate is always zero for ship B).

Now using the distance formula in two dimensions, the squared distance between two points (x_1, y_1) and (x_2, y_2) is given by:

$$D^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Now in our case, considering the quantities as functions of time,

$$D(t)^2 = (A(t) - 0)^2 + (0 - B(t))^2 = A(t)^2 + B(t)^2$$

Differentiating implicitly with respect to time, we get

$$2D(t)D'(t) = 2A(t)A'(t) + 2B(t)B'(t)$$

which we can simplify to

$$D'(t) = \frac{A(t)A'(t) + B(t)B'(t)}{D(t)}$$

At 4 o'clock,

$$A(t) = 35 \cdot 4 - 150 = -10 \quad B(t) = 25 \cdot 4 = 100$$

$$D(t) = \sqrt{A(t)^2 + B(t)^2} = \sqrt{10100} = 100.5$$

$$A'(t) = 35 \quad B'(t) = 25$$

We can substitute these values into our expression for $D'(t)$ to get:

$$D'(t) = \frac{A(t)A'(t) + B(t)B'(t)}{D(t)} = \frac{-10 \cdot 35 + 100 \cdot 25}{100.5} = 21.39 \text{ km/hr}$$

6 An *even function* is defined as a function for which

$$f(-x) = f(x)$$

Show that the **derivative** of $\sinh x$ is an even function.

Solution: First we have to find the derivative of $\sinh x$. If you remember that these derivatives follow a pattern similar to that of the circular trigonometric functions, you can just write down $\cosh x$ for the derivative. Otherwise, you can derive this result from the definition of $\sinh x$:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

so

$$\frac{d}{dx} \sinh x = \frac{e^x - e^{-x}(-1)}{2} = \frac{e^x + e^{-x}}{2} = \cosh x$$

Now start with $\cosh -x$ and see if we can transform it into $\cosh x$:

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^x}{2} = \cosh x$$

7 A cell culture of a certain bacterium has an initial population of 20 cells. Each cell divides into two cells every 20 minutes.

a) Find the growth constant k

b) Find an expression for the number of cells in the culture after t hours

c) Find the number of cells after 8 hours

Solution:

a) Each cell divides every 20 minutes, so the doubling time (in **hours**) will be $t_d = 1/3$ hours, and to find k we solve the equation

$$2P_0 = P_0 e^{kt_d} \quad \Rightarrow \quad 2 = e^{kt_d} \quad \Rightarrow \quad \ln 2 = k \cdot t_d$$

so

$$k = \frac{\ln 2}{t_d} = \frac{\ln 2}{1/3} = 3\ln 2 = 2.079$$

b) We are given $P_0 = 20$ so the population after t hours will be

$$P(t) = P_0 e^{kt} = 20e^{2.079t}$$

c) Using the result of part b),

$$P(8) = 20e^{2.079 \cdot 8} = 20e^{16.63} \approx 16,718,000$$

8 A function is defined piecewise by

$$f(x) = \begin{cases} a \cdot \arctan x + b & x < 0 \\ -2x^3 + 2x + 1 & x \geq 0 \end{cases}$$

What values of a and b (if any) make:

a) f continuous at $x = 0$?
b) f', the **derivative** of f continuous at $x = 0$?
c) f', the **derivative** of f continuous and *differentiable* at $x = 0$?

Solution: This is a piecewise function with two different rules of assignment. Loosely speaking, if you can find values of a and b that make the function values and the first two drivatives of both rules of assignment match at $x = 0$, then you have found a solution to the problem. You can easily verify that choosing $a = 2$ and $b = 1$ will ensure that f is continuous at zero and f' is continuous and differentiable at zero.

The solution presented here derives this solution, but with each step rigorously justified (hence the lengthy argument). It is not necessary to include all of the justifications to receive credit for this problem, but if you can follow this line of argument and understand each step in the process, you can rightly say that you have a thorough understanding of continuity and differentiability as presented in this course.

In order for f to be continuous at $x = 0$, by definition it must be true that

$$\lim_{x\to 0} f(x) = f(0) = a \arctan 0 + b = b$$

From the definition of f, we can evaluate $f(0)$,

$$f(0) = a \arctan 0 + b = b$$

The existence of $\lim_{x\to 0} f(x)$ implies the existence of the left and right hand limits, and these must equal the two-sided limit:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0} f(x) = \lim_{x\to 0^+} f(x)$$

The right hand limit has to be $f(0) = b$, so the left hand limit must also be b:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -2x^3 + 2x + 1 = b$$

But for $x < 0$, $f(x)$ is a polynomial, which is continuous for all x, so the left hand limit as x approaches zero must be equal to the value of the polynomial $-2x^3 + 2x + 1$ when $x = 0$, and we can write

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -2x^3 + 2x + 1 = -2 \cdot 0^3 + 2 \cdot 0 + 1 = b$$

so b has to be equal to 1. This is all that we require to make f continuous at zero. a plays no role in determining the value of f at $x = 0$ because $\arctan 0 = 0$.

b) In order for the first derivative to be continuous at $x = 0$, by the definition of continuity we must have:

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0+} f'(x) = \lim_{x\to 0} f'(x) = f'(0)$$

To the right of zero,

$$f(x) = a \arctan x + b \quad \text{so} \quad f'(x) = \frac{a}{1+x^2}$$

The function $y = 1/(1+x^2)$ is defined and continuous for all real numbers x, so we can evaluate the right hand limit as $x \to 0$ by direct substitution:

$$\lim_{x\to 0^+} \frac{a}{1+x^2} = \frac{a}{1+0}^{2} = a$$

so in order for f' to be continuous at 0, it must be that $f'(0) = a$. To the right of zero, f is a polynomial and

$$f'(x) = -6x^2 + 2 \quad x < 0$$

Again, for $f'(x)$ to be continuous at $x = 0$ the right hand limit must equal $f'(0) = a$, so

$$\lim_{x \to 0^-} -6x^2 + 2 = a$$

Since $y = -6x^2 + 2$ is continuous at $x = 0$, the right hand limit has to be $-6 \cdot 0^2 + 2 = 2$, so $a = 2$.

In conclusion, in order for f' to be continuous at zero, we require $a = 2$ and $b = 1$.

c) Now we consider values of a and b that make f' both continuous and differentiable at $x = 0$. We already know that for f' to be continuous, we need $a = 2$ and $b = 1$, so a and b are already determined and the question is whether these values make f' differentiable at $x = 0$ or not. If they do, we are done. If they do not, the request is impossible.

Other than unusual situations like cusps, a continuous function usually fails to be differentiable because it has a sharp corner, like $y = |x|$ has at $x = 0$. Even a a not-so-sharp corner spoils differentiability: the tangent lines as we approach 0 from the left and right have to converge to exactly the same slope, otherwise the slope of the tangent at $x = 0$ is undefined.

Of course here we are talking about the tangent line to f', whose slope is given by the second derivative f'', so the question boils down to whether or not f'' exists at $x = 0$. Recall that, by definition, $f''(0)$ exists if the limit of the usual difference quotient exists, which in turn requires the existence of left and right hand limits:

$$\lim_{x \to 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0} \frac{f'(x) - f'(0)}{x - 0} = f''(0)$$

To the right of zero, we know that

$$\lim_{x \to 0^-} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^-} \frac{(-6x^2 + 2) - (-6 \cdot 0^2 + 2)}{x - 0} = \lim_{x \to 0^-} -6x = 0$$

To the left of zero,

$$\lim_{x \to 0^+} \frac{f'(x) - f'(0)}{x - 0} = \lim_{x \to 0^+} \frac{\frac{1}{1+x^2} - \frac{1}{1+0^2}}{x - 0} = \lim_{x \to 0^+} \frac{\frac{-x^2}{1+x^2}}{x}$$

$$= \lim_{x\to 0^+} \frac{-x^2}{(1+x^2)\cdot x} = \lim_{x\to 0^+} \frac{-x}{1+x^2} = 0$$

by direct substitution. Now we can say that $f''(0)$ exists since

$$\lim_{x\to 0^-} \frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0^+} \frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0} \frac{f'(x)-f'(0)}{x-0} = 0$$

so f' is differentiable at $x = 0$.

9 Newton's law of gravitation states that the attractive force exerted on a body of mass m by a body of mass M is

$$F = \frac{GMm}{r^2}$$

where G is a constant and r is the distance between them in kilometers.

a) What is the rate of change of the force with respect to r when the distance between the objects is 10km?

b) What is the **average** rate of change of the force with respect to r when the distance between the objects increases from 20km to 50km?

Solution:

a) The equations involve the gravitational constant G, and masses M, and m, but we are not given values for them. Presumably, they are not changing, so we can treat them as constants. We consider force as a function of the distance r and differentiate with respect to r:

$$F(r) = \frac{GMm}{r^2} \quad \text{so} \quad \frac{d}{dr}F(r) = -\frac{2GMm}{r^3}$$

When $r = 10$km, the rate of change of force with respect to r is:

$$\left.\frac{d}{dr}F(r)\right|_{r=10} = -\frac{2GMm}{10^3} = -\frac{2GMm}{1000}$$

b) The average rate of change (per kilometer) of the force as r varies from r_1 to r_2 is the total change in F divided by the total change in r:

$$\Delta F_{avg} = \frac{F(r_2)-F(r_1)}{r_2-r_1}$$

In this case $r_1 = 20$km and $r_2 = 50$km, so

$$\Delta F_{avg} = \frac{\frac{GMm}{50^2} - \frac{GMm}{20^2}}{50-20} = (GMm)\frac{\frac{1}{50^2}-\frac{1}{20^2}}{30}$$

$$= -.00007(GMm)$$

10 Find the equation of the line tangent to the curve

$$f(x) = e^x \left(\frac{x^2+2}{\sqrt{x}} \right)$$

at $x = 1$.

Solution: This problem is much easier if you rewrite the second factor as a sum of powers of x:

$$\frac{x^2+2}{\sqrt{x}} = x^{3/2} + 2x^{-1/2}$$

so

$$f(x) = e^x x^{3/2} + e^x x^{-1/2}$$

and we can just apply the product rule to each term separately to get:

$$f'(x) = e^x x^{3/2} + e^x \left(\frac{3}{2} x^{1/2} \right) + e^x x^{-1/2} + e^x \left(-\frac{1}{2} x^{-3/2} \right)$$

This can be similified a bit by collecting the e^x terms,

$$f'(x) = e^x \left[x^{3/2} + \left(\frac{3}{2} x^{1/2} \right) + x^{-1/2} + \left(-\frac{1}{2} x^{-3/2} \right) \right]$$

or, written with radicals instead of fractional exponents,

$$f'(x) = e^x \left[\sqrt{x^3} + \frac{3\sqrt{x}}{2} + \frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{x^3}} \right]$$

(any of these three forms, or an equivalent expression, would be an acceptable answer)