MA125 Exam 1

Name:

1) The position at time t of a particle moving along a straight line is given by

$$f(t) = t^2 + at + 2$$

where a is a constant chosen to make the particle's **average velocity** from $t = 1$ to $t = 3$ equal to 5.

a) What is the value of a?

b) What is the instantaneous velocity at $t = 1$?

$$v_{avg} = \frac{f(3)-f(1)}{3-1} = \frac{(3^2+3a+2)-(1^2+a+2)}{2} = \frac{8+2a}{2} = 4+a$$

$$4+a=5 \quad a=1$$

$$v_{inst} = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \frac{((1+h)^2+1(1+h)+2)-(1^2+1+2)}{h}$$

$$= \frac{1+2h+h^2+1+h+2-1-1-2}{h} = \frac{3h+h^2}{h} = 3+h$$

$v_{inst} = 3$

$$\lim_{h\to 0} 3+h = 3$$

2) A function $f(x)$ is defined by

$$f(x) = e^{(x+2)} - 5$$

Find the set of real numbers on which the **INVERSE** of f is continuous.

$y = e^{(x+2)} - 5$

$y+5 = e^{x+2}$

$\ln(y+5) = x+2$

$\ln(y+5) - 2 = x$

interchange x and y:

$y = \ln(x+5) - 2$

$\ln x$ is continuous on its domain, which is $(0,\infty)$.

The argument $(x+5)$ must be >0,

so $x+5>0$

$\Rightarrow x>-5$ $\quad f$ is continuous on $(-5,\infty)$

3) Let

$$f(x) = \frac{\tan x}{\sqrt{4 - x^2}}$$

Describe the set of all values (if any exist) for which $f(x)$ is continuous. Briefly explain how you determined your answer.

$\tan x$ is continuous on its domain, which is all real numbers except $\{\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots\}$ and $\{-\frac{\pi}{2}, -\frac{3\pi}{2}, -\frac{5\pi}{2}, \ldots\}$

(i.e., $\tan x = \frac{\sin x}{\cos x}$, any value for which $\cos x = 0$ must be excluded)

$\sqrt{4-x^2}$ is continuous on $[-2, 2]$ (where $4 - x^2 \geq 0 \Rightarrow x^2 \leq 4$)

The domain of f is $[-2, 2]$ excluding $\frac{\pi}{2}$ and $-\frac{\pi}{2}$:

$[-2, -\frac{\pi}{2}) \cup (-\frac{\pi}{2}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, 2]$

4) Use the intermediate value theorem to prove that the equation

$$f(x) = \sqrt{x^2 + x + 1} - \sqrt{x^2 - x + 1} = 0$$

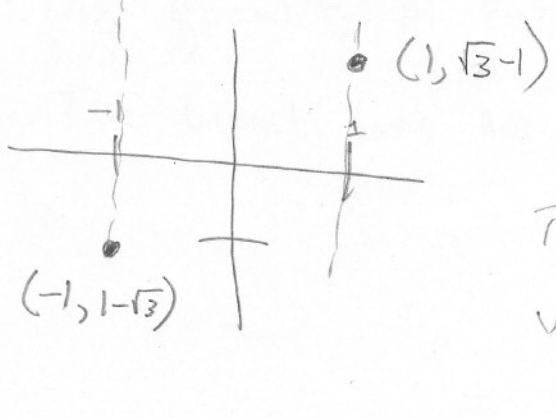
has at least one real root. You should include a justification that the conditions under which the theorem applies are satisfied.

$f(-1) = \sqrt{(-1)^2 - 1 + 1} - \sqrt{(-1)^2 - (-1) + 1}$

$= \sqrt{1 - 1 + 1} - \sqrt{1 + 1 + 1} = 1 - \sqrt{3} < 0$

$f(1) = \sqrt{1^2 + 1 + 1} - \sqrt{1^2 - 1 + 1} = \sqrt{3} - 1 > 0$

Since f is continuous on $[-1, 1]$, and $f(a) \neq f(b)$,



The IVT guarantees that there is a c in $[-1, 1]$ with $f(c) = 0$.

5) A function is defined piecewise by:

$$\begin{cases} k & \text{if } x = 0 \\ f(x) & \text{if } x \neq 0 \end{cases}$$

If f has the property that

$$\frac{x^2}{\sqrt{x^2+4}-2} \leq f(x) \leq 2\sqrt{2}\left(\frac{2-x}{\sqrt{2}-\sqrt{x}}\right) \qquad x \in [-1, 1],$$

determine the value of k that makes f continuous at $x = 0$.

for f to be continuous at $x=0$, $\lim_{x\to 0} f(x) = f(0)$

$$\lim_{x\to 0} \frac{x^2}{\sqrt{x^2+4}-2} = \lim_{x\to 0} \frac{x^2}{(\sqrt{x^2+4}-2)} \frac{(\sqrt{x^2+4}+2)}{(\sqrt{x^2+4}+2)} = \lim_{x\to 0} \frac{x^2(\sqrt{x^2+4}+2)}{x^2+4-4}$$

$$= \lim_{x\to 0} \sqrt{x^2+4}+2 = 4 \quad \left(\text{direct substitution}\right)$$

$$\lim_{x\to 0} 2\sqrt{2}\left(\frac{2-x}{\sqrt{2}-\sqrt{x}}\right) = 2\sqrt{2}\left(\frac{2}{\sqrt{2}}\right) = 4$$

By direct substitution.

By the Squeeze Theorem,

$$\lim_{x\to 0} \frac{x^2}{\sqrt{x^2+4}-2} \leq \lim_{x\to 0} f(x) \leq \lim_{x\to 0} 2\sqrt{2}\left(\frac{2-x}{\sqrt{2}-\sqrt{x}}\right)$$

$$4 \leq \lim f(x) \leq 4$$

Since $\lim_{x\to 0} f(x) = 4$, $f(0)$ must be 4 if f is continuous.

6) Determine whether the following limit exists. If it exists, find its value.

$$\lim_{x\to 1} \ln\left(\frac{2x^2+x+3}{2x^2-x-1}\right)$$

$$= \lim_{x\to 1} \ln\left(\frac{2x^2+x+3}{(2x+1)(x-1)}\right)$$

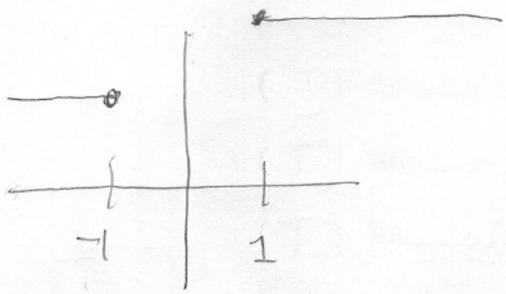
The quantity in parenthesis $\to \infty$ as $x \to 1$.

The Limit does not exist.

7 A function f is defined piecewise by:

$$f(x) = \begin{cases} 2 & x \leq -1 \\ a\ln(x^2+x+1)+b & -1 < x < 1 \\ 3 & x \geq 1 \end{cases}$$

For what values of a and b is f continuous on $[-5, 5]$?



For f to be continuous on $[-5,5]$, we need:

$\lim_{x \to -1^+} f(x) = 2$ and $\lim_{x \to 1^-} f(x) = 3$

$a\ln(x^2+x+1)+b$ is continuous for all x.

By direct substitution,

$$\lim_{x \to -1^+} f(x) = a\ln(1^2-1+1)+b = 2$$
$$= a\ln 1 + b = 2$$
$$a \cdot 0 + b = 2$$
$$b = 2$$

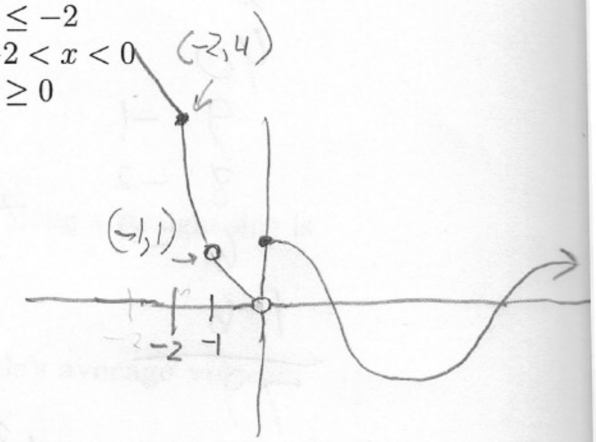
$$\lim_{x \to 1^-} f(x) = a\ln(1^2+1+1)+2 = 3$$
$$a\ln 3 + 2 = 3$$
$$a\ln 3 = 1$$
$$a = \frac{1}{\ln 3}$$

(OVER)

$$f(x) = \begin{cases} 2 & x \leq -1 \\ \frac{\ln(x^2+x+1)}{\ln 3}+2 & -1 < x < 1 \\ 3 & x \geq 1 \end{cases}$$

8 A function $f(x)$ is defined piecewise by the following rule of assignment:

$$f(x) = \begin{cases} 2 - x & \textit{when} \quad x \leq -2 \\ (x^4 - x^2)/(x^2 - 1) & \textit{when} \quad -2 < x < 0 \\ \cos x & \textit{when} \quad x \geq 0 \end{cases}$$



Which of the following statements are true?

(T) $\lim_{x \to -2^-} f(x)$ exists

(T) $\lim_{x \to -2^+} f(x)$ exists

(T) $\lim_{x \to -2} f(x)$ exists

(T) $f(x)$ is continuous from the left at $x = -2$

(T) $f(x)$ is continuous from the right at $x = -2$

(T) $f(x)$ is continuous at $x = -2$

(T) $\lim_{x \to -1^-} f(x)$ exists

(T) $\lim_{x \to -1^+} f(x)$ exists

(T) $\lim_{x \to -1} f(x)$ exists

(F) $f(x)$ is continuous at $x = -1$

(F) $f(x)$ is continuous from the left at $x = 0$

(T) $f(x)$ is continuous from the right at $x = 0$

(F) $f(x)$ is continuous everywhere except -1 and 0 and +

(T) $f(x)$ is continuous at $x = \pi/2$

(T) $f(x)$ is continuous on the interval $[\pi/2, \infty)$

(T) $f(x)$ is continuous on the interval $\cancel{(}\infty, -1.5]$

$$\frac{(x^4 - x^2)}{x^2 - 1} = \frac{x^2(x^2-1)}{(x^2-1)} = \frac{x^2(x+1)(x-1)}{(x+1)(x-1)} = x^2 \quad \text{if } x \neq \pm 1$$