MA125 Exam 2

Name: KEY

1) Find the derivative of the following function and its domain:

$$h(x) = 1 - \sin x \cdot \sec x$$

Method 1:

$f(x) = \sin x$

$f'(x) = \cos x$

$g(x) = \sec x$

$g'(x) = \sec x \tan x$

$$h'(x) = 0 - [fg' + gf']$$
$$= -\sin x \sec x \tan x - \sec x \cos x$$
$$= -\tan^2 x - 1$$
$$= -\sec^2 x$$

Method 2:

write $\sin x \cdot \sec x$ as $\tan x$. Then:

$$h'(x) = 0 - \sec^2 x$$

2) Find the slope of the line tangent to the curve

$$f(\theta) = \frac{1+\sin\theta}{1+\cos\theta}$$

at $\theta = 3\pi/2$.

$u = 1 + \sin\theta$

$v = 1 + \cos\theta$

$u' = \cos\theta$

$v' = -\sin\theta$

$$f'(\theta) = \frac{vu' - uv'}{v^2} = \frac{\cos\theta + \cos^2\theta + \sin\theta + \sin^2\theta}{(1+\cos\theta)^2}$$
$$= \frac{1 + \cos\theta + \sin\theta}{(1+\cos\theta)^2}$$

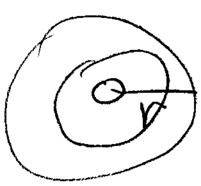
$$f'\left(\frac{3\pi}{2}\right) = \frac{1 + \cos\frac{3\pi}{2} + \sin\frac{3\pi}{2}}{\left(1 + \cos\frac{3\pi}{2}\right)^2} = \frac{1 + 0 + -1}{(1+0)^2} = 0$$

3 A stone is dropped into a still pond creating a circular wave that expands away from the center at a rate of 50 centimeters per second. Find the rate of change of the **area** inside the circle after:

a) 1 second
b) 3 seconds
c) 5 seconds

Think of this as: $A(t) = \pi[r(t)]^2$.

differentiate with respect to time:



$A = \pi r^2$

$\frac{dr}{dt} = 50$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 100\pi r(t)$$

a) $\frac{dA}{dt} = 100\pi \cdot 1 = 100\pi$

b) $\frac{dA}{dt} = 100\pi \cdot 3 = 300\pi$

c) $\frac{dA}{dt} = 100\pi \cdot 5 = 500\pi$

4) Find the derivative of the following function directly from the definition of the derivative as a limit:

$$f(x) = 1 + \sqrt{x+3}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1+\sqrt{x+h+3} - (1+\sqrt{x+3})}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h+3} - \sqrt{x+3}}{h} = \lim_{h\to 0} \frac{(\sqrt{x+h+3} - \sqrt{x+3})}{h}\frac{(\sqrt{x+h+3} + \sqrt{x+3})}{(\sqrt{x+h+3} + \sqrt{x+3})}$$

$$= \lim_{h\to 0} \frac{(x+h+3)-(x+3)}{h(\sqrt{x+h+3} + \sqrt{x+3})} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+3} + \sqrt{x+3})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h+3} + \sqrt{x+3}}$$

$$= \frac{1}{2\sqrt{x+3}}$$

5) Find the horizontal and vertical asymptotes of the function

$$g(x) = \frac{2x^2+x-1}{x^2+x-2}$$

$g(x) = \dfrac{2x^2+x-1}{(x+2)(x-1)}$ ← denominator is zero at $x=1$ and $x=-2$
numerator is not zero ⇒ vertical Asymptotes at $x=1$ and $x=-2$

now write $g(x)$ as:

$g(x) = \dfrac{2+\frac{1}{x}-\frac{1}{x^2}}{1+\frac{1}{x}-\frac{2}{x^2}}$ as $x \to \pm\infty$, This ratio tends to $\frac{2}{1} = 2$

∴ $y=2$ is a horizontal asymptote.

6) Find the equation of the line tangent at $x = 0$ to the function $(f \cdot g)(x)$ given that

$$f(x) = 3x^2+1 \quad \text{and} \quad g(x) = \cos x$$

The tangent line to $h(x)$ at $x=a$ is always:

Let $h(x) = (f \cdot g)(x)$ $\qquad$ $y - h(a) = h'(a)(x-a)$

Then $h'(x) = f'g + g'f$ by The product rule.

The tangent line at $x=a$ is

$$y - (3a^2+1)(\cos a) = [(3a^2+1)(-\sin a) + (\cos a)(6a)](x-a)$$

When $a=0$, This becomes

$y - 1 = 0(x-0)$ or $y = 1$

7) A particle moves along a straight line. If its position after t seconds is given by

$$s = 4t^3 - 9t^2 + 6t + 2$$

a) What is the particle's velocity at time $t = 0$?
b) What is the particle's acceleration at time $t = 1$?
c) At what time(s), if any, is the particle at rest?
d) At what time(s), if any, is the particle neither accelerating nor decelerating?

a) Velocity $= \frac{ds}{dt} = 12t^2 - 18t + 6$ $\quad V(0) = +6$

b) acceleration $= \frac{d^2s}{dt^2} = \frac{dv}{dt} = 24t - 18$ $\quad A(1) = +6$

c) at rest when $V(t) = 0 = 6(2t^2 - 3t + 1) = 6(2t-1)(t-1)$

zero at $t = 1/2,\ t = 1$

d) not accelerating if $A(t) = 24t - 18 = 0 \Rightarrow t = 3/4$

8) Find the derivative of

$$f(x) = \frac{1 + \sin x}{1 + e^x}$$

What is the equation of the line tangent to $f(x)$ at $x = 0$?

Quotient rule:
$u(x) = 1 + \sin x$
$u'(x) = \cos x$
$v(x) = 1 + e^x$
$v'(x) = e^x$

$$f'(x) = \frac{vu' - uv'}{v^2} = \frac{(1+e^x)\cos x - (1+\sin x)e^x}{(1+e^x)^2}$$

Rather than simplify, just plug in $x=0$ and evaluate:

$$f'(0) = \frac{(1+e^0)(\cos 0) - (1+0)e^0}{(1+e^0)^2} = \frac{2 \cdot 1 - 1 \cdot 1}{2^2} = \frac{1}{4}$$

$f(0) = \frac{1}{2}$

$\Rightarrow y - \frac{1}{2} = \frac{1}{4}(x - 0)$ $\quad \boxed{y = \frac{1}{4}x + \frac{1}{2}}$

9) A function is defined piecewise by

$$f(x) = \begin{cases} x^2 + bx + c & x < 0 \\ 1 + \sin x & x \geq 0 \end{cases}$$

a) What value of c makes f continuous at $x = 0$?
b) What values of b and c make f continuous and differentiable at $x = 0$?

a) To be continuous, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 1 + \sin 0 = 1$

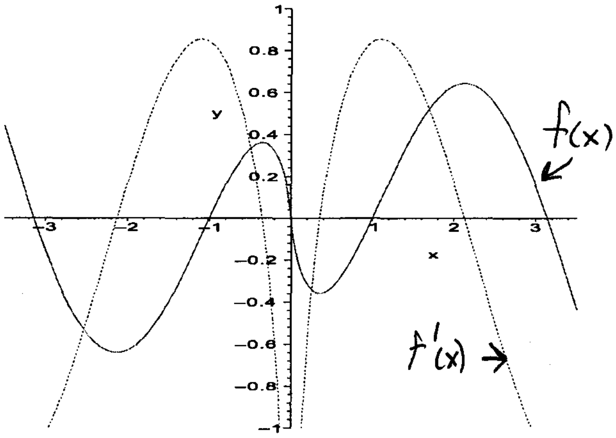
$\lim_{x \to 0^-} x^2 + bx + c = c$, so $c = 1$

b) To be differentiable, the slopes must match as we approach zero from the left and right.

So, $\frac{d}{dx}(x^2 + bx + c)$ must equal $\frac{d}{dx}(1 + \sin x)$ when $x = 0$

$\Rightarrow b = 1$

10) Given the following graphs of $f(x)$ and $f'(x)$,



a) Identify which curve is $f(x)$ and which is $f'(x)$.

f(x) is the solid line

b) Identify the intervals on which $f(x)$ is increasing

f(x) is increasing when $f'(x) > 0$: $(-2.1, -0.5) \cup (0.5, 2.1)$

c) Identify the intervals on which $f(x)$ is decreasing

f(x) is decreasing when $f'(x) < 0$: $(-3.4, -2.1) \cup (-0.5, 0) \cup (0, 0.5) \cup (2.1, 3.4)$

d) Identify the intervals on which the **second derivative** $f''(x)$ is positive

Second derivative positive $\Rightarrow$ f concave up: $(-3.4, -1) \cup (0, 1)$

e) Identify the intervals on which the **second derivative** $f''(x)$ is negative

f'' negative $\Rightarrow$ f concave down: $(-1, 0) \cup (1, 3.4)$

f) Identify the point(s) at which f is not differentiable

f appears to be not differentiable at $x=0$, because the tangent line is vertical.