MA125 Exam 1

Name:

KEY

1) A function $f(x)$ is defined by

$$f(x) = \frac{x+2}{2x-5}$$

Find the set of real numbers on which the **INVERSE** of f is continuous (i.e., find $\{a \; : \; f^{-1}(x) \quad \text{is continuous at } x = a\}$).

$y = \frac{x+2}{2x-5}$

$y(2x-5) = x+2$

$2xy - 5y = x+2$

$2xy - x = 5y+2$

$x(2y-1) = 5y+2$

$x = (5y+2)/(2y-1)$

Interchange x and y:

$y = f^{-1}(x) = \left(\frac{5x+2}{2x-1}\right)$

Rational function is continuous on its domain, $(-\infty, .5) \cup (.5, \infty)$ (everywhere except .5)

2) A particle is moving along a straight line and its position at time t is given by

$$f(t) = \frac{2t+5}{4t+1}$$

what is the particle's **average velocity** from $t = 0$ to $t = 0.5$?

$$V_{avg} = \frac{f(0.5) - f(0)}{.5 - 0} = \frac{\frac{2(.5)+5}{4(.5)+1} - \frac{2(0)+5}{4(0)+1}}{.5} = \frac{\frac{6}{3} - \frac{5}{1}}{.5} = \frac{-3}{.5} = -6$$

3) Let

$$f(x) = \frac{\ln(x^2+6)}{x^2-16}$$

Describe the set of all values (if any exist) for which $f(x)$ is continuous on the interval $[-10, 10]$.

$\ln x$ is continuous on its domain, $\{x : x > 0\} = (0, \infty)$

x^2+6 is positive for any x, so the numerator is continuous for all real x.

$x^2 - 16 = 0$ when $x = \pm 4$, so the rational function

$\frac{1}{x^2-16}$ is continuous everywhere except $x = \pm 4$. On $[-10, 10]$,

f is continuous everywhere except ± 4, or on $[-10, -4) \cup (-4, 4) \cup (4, 10]$

4) Use the intermediate value theorem to prove that the equation

$$e^x - e^{-x} = 2$$

has at least one real root. You should include a justification that the conditions under which the theorem applies are satisfied.

Let $f(x) = e^x - e^{-x} - 2$. Since e^x is continuous for all real x, $e^x - e^{-x} - 2$ is continuous for all real x. If we can find an interval $[a,b]$ with $f(a) < 0$ and $f(b) > 0$, the Intermediate Value Theorem guarantes $f(c) = 0$ for some $c \in [a,b)$. $f(0) = -2$

$f(4) = e^4 - e^{-4} - 2 > 0$

choose $[-2, 4]$

5) A function $f(x)$ has the following property:

$$\frac{x^2 - 6x}{x^2 - 10x + 24} \leq f(x) \leq \frac{x}{8 - x} \qquad x \in [5, 7]$$

Find $x \to 6$, if it can be found using this information.

$\frac{x^2 - 6x}{x^2 - 10x + 24} = \frac{x(x-6)}{(x-6)(x-4)} = \frac{x}{x-4}$ when $x \neq 6$, so $\lim_{x \to 6} \frac{x^2 - 6x}{x^2 - 10x + 24} = \lim_{x \to 6} \frac{x}{x-4} = 3$

$\lim_{x \to 6} \frac{x}{8-x} = 3$ by direct substitution, so by the Squeeze Theorem,

$3 \leq \lim_{x \to 6} f(x) \leq 3$ and $\lim_{x \to 6} f(x) = 3$

6) Suppose

$$f(x) = \sqrt{x} \quad \text{and} \quad g(x) = \frac{f(x) - f(3)}{x - 3}$$

Find $\lim_{x \to 3} g(x)$ if the limit exists.

$g(x) = \frac{f(x) - f(3)}{x - 3} = \frac{\sqrt{x} - \sqrt{3}}{x - 3} = \frac{\sqrt{x} - \sqrt{3}}{x - 3}\left(\frac{\sqrt{x} + \sqrt{3}}{\sqrt{x} + \sqrt{3}}\right) = \frac{x - 3}{(x-3)(\sqrt{x} + \sqrt{3})}$

$= \frac{1}{\sqrt{x} + \sqrt{3}}$ if $x \neq 3$

By direct substitution, $\lim_{x \to 3} \left(\frac{1}{\sqrt{x} + \sqrt{3}}\right) = \frac{1}{\sqrt{3} + \sqrt{3}} = \frac{1}{2\sqrt{3}}$

So

$\lim_{x \to 3} \frac{\sqrt{x} - \sqrt{3}}{x - 3} = \frac{1}{2\sqrt{3}}$

7 A function f is defined piecewise by:

$$f(x) = \begin{cases} 2 & x < 0 \\ ab^x & 0 \le x \le 2 \\ \frac{2}{9} & x > 2 \end{cases}$$

For what values of a and b is f continuous on $(-\infty, \infty)$?

If f is continuous on $(-\infty,\infty)$, f is continuous at $x=0$, which means: ~~the limit~~

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0),$$

Since $\lim_{x \to 0^-} = 2$, $f(0) = 2$

By a similar argument, $f(2) = 2/9$. So, now the problem is to find ab^x given that it passes through $(0,2)$ and $(2, 2/9)$.

$f(0) = ab^0 = a \cdot 1 = 2$ so $a = 2$

$f(2) = 2b^2 = \frac{2}{9}$ so $b^2 = \frac{1}{9}$ and $b = \frac{1}{3}$.

(OVER)

8 A function $f(x)$ is defined piecewise by the following rule of assignment:

$$f(x) = \begin{cases} 0 & when \;\; x \le 0 \\ e^{x-1} - 1 & when \;\; 0 < x < 1 \\ \ln x & when \;\; x \ge 1 \end{cases}$$

Which of the following statements are true?

(T) $\lim_{x\to 0^-} f(x)$ exists

(T) $\lim_{x\to 0^+} f(x)$ exists

(F) $\lim_{x\to 0} f(x)$ exists

(T) $f(x)$ is continuous from the left at $x = 0$

(F) $f(x)$ is continuous from the right at $x = 0$

(F) $f(x)$ is continuous at $x = 0$

(T) $\lim_{x\to 1^-} f(x)$ exists

(T) $\lim_{x\to 1^+} f(x)$ exists

(T) $\lim_{x\to 1} f(x)$ exists

(T) $f(x)$ is continuous from the left at $x = 1$

(T) $f(x)$ is continuous from the right at $x = 1$

(T) $f(x)$ is continuous at $x = 1$

(T) $f(x)$ is continuous at $x = -2$

(T) $f(x)$ is continuous on the interval $(-\infty, -2]$

(T) $f(x)$ is continuous on the interval $[\frac{1}{2}, \infty)$

(F) $f(x)$ is continuous on the interval $[-1, 1]$